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INVESTIGATING SOCIAL SCIENCE STUDENTS' UNDERSTANDING OF LIMITS THROUGH THE LENS OF THE PROCEPT THEORY

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ABSTRACT

The study reported in this paper sought to investigate how social science students understand the idea of limit with regard to the use of its symbolism. Sixty first year university students in the social sciences acted as the sample of the study. An adapted procept theory was used to analyse data obtained from these students through their solution to tasks on limit and explanations on their thinking and solution processes. Data analysis indicated that some students understood the limit symbolism $\lim_{x \to a} f(x)$ *to be a procept while others did not. When solving the mathematical tasks, students' difficulties emanated from: (i) their inability to coordinate the two processes,* $f(x) \to L$ *and* $x \to a$*, or* $f(x) \to L$ *and* $x \to \infty$ *(ii) the proper use of the limit operator,* $\lim_{x \to a}$*, and (iii) inability to realise that the simplification has led to the same response as they could not see the relationship between their working and the results. This resulted in misalignment between their reasoning and their choice of answers where justification was required. The results also show that limits at infinity were more problematic than those of the form* $f(x) \to L$ *as* $x \to a$*, where a is a constant. Students' choice of method used depended mostly on how much efficient the method was in terms of saving time and not really on promoting understanding.*

Keywords: procedure, process, concept, procept, limits, mathematical symbols

INTRODUCTION

Research in mathematics and mathematics education has shown that success in mathematics requires much more than being good at carrying out algorithmic or mechanistic procedures that lead to the solution of the mathematics problem at hand (García-García & Dolores-Flores, 2021; Mellor, Clark & Essien, 2018). Such algorithmic processes need to be complemented by some kind of a holistic grasp of the context (Essien, 2021; Tall & Thomas, 1991). In the mathematics context, symbols are important in helping to mediate the cognitive processes around the concept at hand, and also enable performing operations on them (Güçler, 2014). This is because symbols are tools with which to represent concepts (objects) and processes (Gray & Tall, 1994; Güçler, 2014). Through the use of symbols, many students however write procedures which are hardly related to their conceptual meaning in solving mathematical problems (Tall & Thomas, 1991). This sequential (procedural) process of responding to questions poses a lot of problems for a variety of reasons: (i) an incorrect answer may be obtained and failed to be recognised; and (ii) there may be difficulties in answering questions which require interpretation. To address this problem, teachers need to

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unpack the meanings inherent in symbols to enhance mathematical communication in the classrooms (Güçler, 2014). Hence, the use and interpretation of symbols are not necessarily without problems, as in some contexts in mathematics, in particular limits for example, the same symbol may represent both the concept and the process (Tall & Thomas, 1991; Güçler, 2014). As an example, the notation $\lim_{x \to a} f(x)$ represents both the process of *tending to a limit* and the concept of the *value of the limit.* This dual nature of mathematical symbols as a process and a concept is referred to by Tall and Thomas (1991) and Gray and Tall (1994) as a *procept*. In other words, this dualism shows the role of symbols in mathematics as tools that allow the human mind to switch effortlessly from "concepts to think about" to "processes to solve problems" (Tall, n.d. p.1). Sfard (1991) refers to this dualism structurally - as objects, and operationally - as processes. According to Sfard (1991):

> Seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing - a static structure, existing somewhere in space and time. It also means being able to recognize the idea "at a glance" and to manipulate it as a whole, without going into details.… In contrast, interpreting a notion as a process implies regarding it as a potential rather than actual entity, which comes into existence upon request in a sequence of actions (p.5).

Another type of limit referred to as *limit at infinity*, denoted by $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to \infty} f(x)$, may also be viewed as a procept (Gray & Tall, 1992). But as Gray and Tall (1992 cautions, it is important to note that not all mathematical concepts can be viewed as procepts.

The empirical work that has been done to investigate students' understanding of limits were mostly in the fields of natural sciences (Moru, 2009; Maharaj, 2010; Güçler, 2014; Jones, 2015) engineering and mathematics (Güçler, 2014; Jones, 2015), and Technology (Cottrill et al., 1996; Güçler, 2014). We did not come across any studies on limits in the social sciences where students do not take mathematics as their major subject. To address this concern, the purpose of the reported study was to investigate how social science students understand and respond to problems on limits represented algebraically. This is because this is the context in which symbols seem to show their dual nature (Sfard, 1991; Gray and Tall, 1994; Güçler, 2014; Jones 2015). As Güçler (2014) contends, the issue for learners is not whether they consider limits as processes or objects but whether they can consider limits as also depending on the mathematical context. Such a flexible utilisation of the limit notation requires the understanding of the concept of limit together with the processes associated with it (Gray & Tall, 1994).

As the title suggests, the sample of the study is the social science students. These students are required to understand the idea of limit because of its applicability in their area of study. In marginal analysis, social science students engage in ideas such as marginal supply, marginal demand, marginal propensity to save and to consume and many more. The mode of representation of concepts in such contexts make use of symbols and the most prominent symbols are that of the idea of limit. Symbols representing limits may be seen as concepts, processes or procepts. Understanding these conceptions enable students to interpret the obtained numerical or algebraic results in solving problems within their field with great depth and accuracy. Thus, it is of absolute necessity for students' work or understanding to be analysed using the Procept Theory framework.

The following research questions emanated from the purpose of the study:

1. How do Social Science University students understand mathematical symbols representing the idea of limit?
2. What methods or techniques do they employ in solving mathematical tasks on limits?
3. What justification(s) (if any) do students give for a preferred method in solving tasks over the other?

We believe that the findings of the study will contribute to the literature on how symbols in calculus, especially in limits, are understood by students who are non-mathematics majors. The findings will also show why students prefer certain methods or procedures to others. The questions that are important to us as far as research questions 2 and 3 are concerned are: Is the procedure or method chosen on the basis of its importance in improving understanding? Are the procedures chosen based on its efficiency? Is there a valid reason for a preferred procedure/method?

THE TEACHING AND LEARNING OF LIMITS

The idea of limits was not an explicit construct in its early development but since the 19^{th} century, it has been considered a central concept in calculus (Viirman, Vivier & Monaghan, 2022). In teaching, both the dynamic and the informal language of limits was used. The static language of limits involving the $\varepsilon - \delta$ notation only became prominent after the 19^{th} century (ibid.). Since the current study focuses on the former language, the studies considered in the literature review will exclude the latter.

Monaghan (1991) found that some students had problems with the limit concept because of the ambiguity inherent in the phrases and terms used in its context (also see Viirman et al., 2022). These include phrases and terms such as: 'tends towards', 'approaches', "close to" and 'limit'. The phrase 'tends towards' may mean either approaching and reaching or approaching without reaching (Cornu, 1991). According to Taback (1975), the word reach may mean being in the neighbourhood of a point or landing on a point. Williams' (1991) study revealed that students perceived the idea of limit as a boundary or something unreachable, the meaning that overlaps with that of Cornu (1991). The phrase 'close to' poses problems as to how close one can be to the point that is being approached. This becomes difficult for students especially when choosing numbers in the neighbourhood of the number that is being approached. Is one allowed to be a tenth, a hundredth or a millionth away? Such skills of making proper choices need the proper understanding of the concept of number as each point or number chosen has to be in the direction of the number that is being approached either from the left-hand side or the right-hand side.

The difficulties that students encounter when dealing with solving problems in limits as Moru (2009) concluded from her study include: (i) the limit does not exist where the function is not defined, (ii) If $f(x)$ is not defined at a point, the functions values tend to infinity, and (iii) that the function value is the limit value. Cottrill et. al. (1996) attribute the main problem of lack of understanding of the limit concept by the students as failure to coordinate the two processes $f(x) \to L$ as $x \to a$. In the same light, Denbel's (2014) study shows that some of the students' misconceptions about the idea limit are that: (i) students think that limits simply entail substituting the value at which the limit is to be found, into the expression, (ii) they often think that limits are only encountered when trying to ascribe a value to a function at a point where was same, (iv) students talk of a limit not being defined at a point, when it is the function that is not defined at the point, and (v) students think only about the manipulative aspects and do not focus on the idea of the limit. With regard to the interpretation of what the limit value is, there is an overlap with the descriptions given by Williams (1991). Some students interpreted the limit as the boundary. This shows how everyday language influences students' understanding of some technical terms in mathematics. Students are not aware that such terms carry a different meaning in the mathematical context as the meaning of words or terms is context bound.

Güçler (2014) conducted a study which addressed the question: "How do one instructor and his students use and think about the limit notation in a beginning-level undergraduate calculus classroom?" (p.251). The findings of the study show that although the instructor differentiated between the process and the product aspects of the limit, students still perceived the limit as a process when using its notation. In the same vein, Jones (2015) conducted a study that sought to investigate the dynamic reasoning used by undergraduate students when thinking about, calculating, and interpreting limits at infinity. The type of infinity considered here is the *potential* and not the *actual* infinity. The potential infinity is taken as a never-ending process while the actual infinity is the existent entity such as the number of elements in the set of integers, for example. In dealing with limits at infinity, in Jones' (2015) study, students are said to have taken ∞ as a quantity that can be substituted for a variable and be manipulated. For example, in finding the limit of $f(x) = \frac{1}{x}$ as x tends to ∞, the symbol ∞ was substituted for x and $\frac{1}{\infty}$ was simplified to zero. Thus, the students did not take into consideration the process of tending to which is the dynamic feel that the potential infinity has. Other students regarded infinity as a big number that could either be negative or positive.

As in the case of Jones (2015), an earlier study by Maharaj (2010), undergraduate science students were asked to find the limit at infinity of rational functions. In addition, students had to find the limit of a rational function not defined at $x = a$. The question on limits at infinity read: the following infinite limit is equal to $\lim_{x\to\infty} \frac{-3x^2+3x-8}{-6x^2+10}$, A) $\frac{-5}{4}$ B) 0 C) ∞ D) $\frac{1}{2}$ E) None of these. Out of 868 students, only 400 (46.1%) chose the correct answer (D). Handling ∞ seems to have been problematic for most students. It was necessary for students to realise that in this case, the procedure is to divide both the numerator and the denominator by the highest power of a variable in the denominator and all the terms whose denominators have the power of x greater zero should have zero as their limit. This would leave the quotient of -3 and -6 which is $\frac{1}{2}$ that no longer depends on x, hence the limit value. Another question in Jones (2015) on finding the limits of rational functions not defined at $x = a$ read: the following $\lim_{x\to 36}\frac{\sqrt{x}-6}{x-36}$ is equal to: A) $\frac{1}{6}$ B) 0 C) -∞ D) $\frac{1}{12}$ E) ∞. Only 254 (29.3%) students chose the correct answer D). Forty one students (4.7%) who got the answer zero, might have substituted 36 in the place of x and worked out the result as zero, although such a quotient was undefined. Factorisation also seems to have been a problem to the majority of students. In particular writing $x - 36$ as a product of the two factors $\sqrt{x} - 6$ and $\sqrt{x} + 6$, was not an obvious simplification for the majority of students. Dealing with a radical sign appears to have been more problematic for students than dealing with the concept of potential infinity (∞). Our study differs from the reported studies here in that student were not only expected to solve the tasks on limits, but were also to justify why their chosen methods were more preferred than others that may have qualified. In this way we managed to access students' thinking about the symbolism used through their written work.

THEORETICAL ORIENTATION: THE NOTION OF PROCEPT

Gray and Tall (1992) proposed the notion of procept (the duality of mathematical symbols as both process and concept) and argue that when students focus mainly on procedures, they may very well be good at computations and succeed in the short term. Such students will, however, lack flexibility of approach that is needed for long-term to succeed in mathematics. This (that is, focus on procedures) they suggest should be complemented by the global view of the concept. Figure 1 shows the spectrum of the stated outcomes of the constructs *pre-procedure, procedure, process, and procept* together with their levels of sophistication from the work of Gray and Tall (2001).

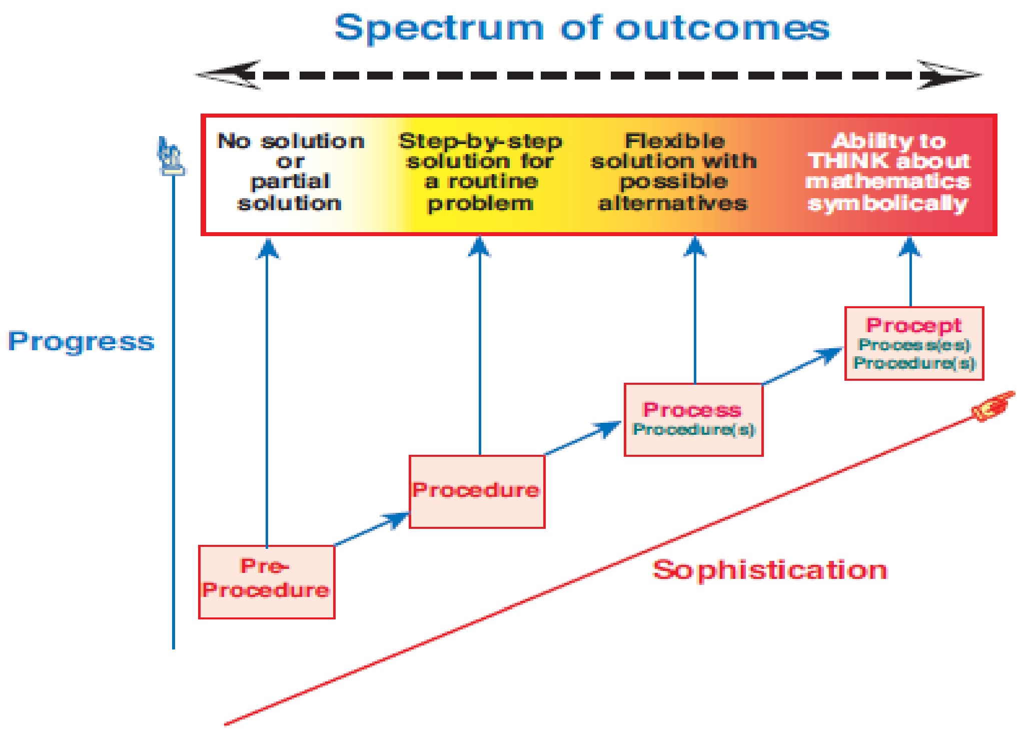

Figure 1. *A spectrum of performance* (Source: Gray & Tall, 2001, p.69)

As highlighted earlier, a *procept* is a combined mental object consisting of both process (series of procedures) and concept in which the same symbolisation is used to denote both the process and the object which is produced by the process (Tall & Gray, 1992). According to Tall (n.d), there are several different ways in which the symbolism is used, namely, (a) a *procedure* which consists of a finite succession of actions and decisions built into a coherent sequence. It is seen essentially as a step-by-step activity with each step triggering the next (or a specific algorithm for implementing a process), (b) *process*, this refers to when the procedure is conceived as a whole and the focus is on input and output rather than the particular procedure used to carry out the process (it may be achieved by *n* procedures and affords the possibility of selecting the most efficient solution in a given context), and (c) a *procept* requires the symbols to be conceived flexibly as processes to do and concepts to think about. Thus, the spectrum of procedure-process-procept is not a classification into disjoint classes because of the interrelations that exist between the three constructs (Gray & Tall, 2001).

Understanding limits using the procept theory

In what follows, we attempt to clarify the terminology used in Figure 1 by means of examples in the context of limits. The Pre-procedure stage is when the student has not responded to the task or has not completed the procedure in solving the task. Procedures in limits include the quotient rule, the product rule, the power rule etc. This requires students to perform routine mechanical steps which may be performed without understanding the concept behind such steps. Processes include $f(x) \to L$ and $x \to a$ which constitute part of informal definition of limit. These processes differ from procedures in that they require the proper understanding of the concepts of "neighbourhood", "number" and "tending to" or "approaching". The choice of numbers in the neighbourhood of x is not done mechanically but with understanding of the number concept. Successive points (numbers) of x that tend to a also have to be chosen with understanding of the number concept. Identifying the limit value requires the coordination of the two processes $f(x) \to L$ and $x \to a$ meaningfully in order to come up with the correct limit value which may not be a number on the table of $f(x)$ values (if done numerically). This is because some calculations will have to be completed mentally, and careful analysis of the number that is being approached from both the left-hand side and the right-hand side by functional values be made through the coordination of $f(x) \to L$ as $x \to a$.

If an expression given is observed in totality (seen as an object) and is broken down through factorisation, for example, a judgement of which procedure to perform still needs some understanding of whether the resulting expression warrants direct substitution or not. The knowledge of how to find the limit of a polynomial function is necessary if such a case arises after the simplification. The restriction of not equating x to a is also necessary in the interpretation of the result of the limit value of the functional value. The notation $\lim_{x \to a} f(x)$ that represents both the process of *tending to a limit* and the concept of the *value of the limit* is a procept. A specific example that could be considered is $\lim_{x \to 2} \frac{x^2-4}{x-2}$, which represents both the process of tending to a limit and the value of that limit, 4. In this case the object, $\frac{x^2-4}{x-2}$, is decomposed by the process of factoring to $\frac{(x-2)(x+2)}{x-2}$ if chosen. Through simplification this leads to finding the expression, $x+2$, provided $x \neq 2$. The new function which is now a polynomial allows for the direct substitution of the number 2 to produce 4. The 4 which is the limit value is obtained by adding 2 to 2. The flexibility of decomposing the function leads to a composition of adding the numbers by following the rules of limits.

The order in which the x values are chosen is also important in infinite limits and the simplification of expressions requires a good judgement as to whether some parts will tend to zero or not by following a procedure of dividing by the highest power of the variable in the denominator. For example, in finding the limit of $\frac{2x^3+x^2+4}{3x^3}$ as $x \to \infty$, dividing each term (numerator and denominator) by x^3 yields $\frac{2+\frac{1}{x}+\frac{4}{x^3}}{3}$ which becomes $\frac{2}{3}$ since the other terms tend to zero as $x \to \infty$ when applying the rules of limits. To some students it may not be so obvious that the terms that lead to zero may be left out and so they may equate the limit of $\frac{2x^3+x^2+4}{3x^3}$ as $x \to \infty$ to the limit of $\frac{2x^3}{3x^3} = \frac{2}{3}$ as $x \to \infty$. This is only done taking into consideration the process of tending to which now excludes the limit operator, $\lim_{x \to a}$. That is, $\lim_{x \to a} \frac{2x^3+x^2+4}{3x^3} = \frac{2+0+0}{3}$ (The zeros should not be written before the process of simplification is completed). These procedures and processes lead to finding the limit value, the object. The given

examples do show that the three constructs procedure, process and procept are indeed not mutually exclusive.
In order to understand why the students viewed the symbols as either processes, objects or procepts, we added an element of letting them justify their answers. This is an element that is lacking in the Procept theory, hence we had to make some changes that would accommodate this element. The adjusted framework also includes where learners have committed errors at various levels of the level of sophistication of the outcomes.

RESEARCH DESIGN

The study followed a case study design which is exploratory in nature. The purpose was to collect qualitative data that will reflect how students understand the symbolism used in limits at different levels of sophistication of the procept theory (procedures, processes or procepts). In addition, students were to solve problems which will reflect their understanding of procedures and processes employed in the computation of the limit values. The methods that were used by students were accompanied by an explanation of why such method was preferred to others that were also appropriate for solving the given task. Very little quantitative aspects that appear in data analyses to create a meaningful whole are complementary to the qualitative results which are more dominant (Starman, 2013).

The sample and data collection techniques

The sample was a group of 60 first year social science students who were registered for Bachelor in Economics and Certificate in Statistics. This group took a Calculus 1 course for non-mathematics majors. They were introduced to calculus for the first time at the university after having completed a course in algebra during the first semester. As indicated earlier, students require to understand the concept of limit in order to tackle problems in their field of specialisation because of its applicability in concepts such as marginal demand, marginal supply, marginal propensity to consume and save, etc. These are some of the concepts which pave way to the understanding of more complex ideas in both micro and macro-economics. Data were collected by letting students solve mathematical tasks. They also needed to provide a justification for the method(s) used. The teaching of the group comprised three lectures per week. The lectures were complemented by a 2-hour tutorial session per week. Each tutorial tested the students' understanding of the content covered in class. In the tutorial sessions, students were divided into smaller groups. Data was collected after nine lectures which were completed in three weeks. This was done during the lecture period. The writing time took one hour under the supervision of the lecturer (the first author) to avoid any discussions that may take place among the students in order to get a more valid data.

The Tasks

The students were given three main tasks to respond to. These were in some cases complemented by sub-questions which required them to reflect on their thoughts as they tackled the tasks or to explain their answers. In Table 1, we provide details of the tasks and the justification for each task in a tabular format.

Table 1. Written Tasks and justification for Task selection

	Tasks	Justification
1	Do the expressions $f(x) \to L$ as $x \to a$ and $\lim_{x \to a} f(x)$ mean one and the same thing? Explain your answer.	This task was asked in order to investigate if students could tell that the symbolism used in the expressions mean one and the same thing even though the first has the dynamic feel while the second seems static. In addition, they were asked to explain their answers to see the type of conceptions that the students had about the symbols in the manner in which they were presented (that is, whether they see a symbol as process, concept or procept)
2	Find $\lim_{x \to 3} \frac{x^3-27}{x-3}$. (i) As you responded to Question 2, what thoughts, questions or ideas came to mind as you were answering the question? (ii) Do you know any other methods besides the one(s) you have used which can be used in answering the question? Mention those other methods (if any). (iii) If yes (in ii above), why did you choose the method/methods you used as opposed to others (iv) If no (that is, if you don't know of any other method), say why you think there is only one method to solving the question.	This was to see if the students would follow the correct procedures in solving the problem step-by-step and/or if they would look at the function in totality and act on it accordingly. Looking at the function in totality required the students to break it down into simpler and manageable components and factoring before substitution as the resulting expression would be a polynomial (and then compose the result to a limit value). The sub-questions were added to see if students chose the methods on the basis of promoting understanding or efficiency in terms of saving time.
3	Is $\lim_{x \to \infty} \frac{8x^3+5}{2x^2+1} = \lim_{x \to \infty} \frac{8x^3}{2x^2}$? (i) If yes, why do you think so? (ii) If no, why do you think so?	For this task, the students were to respond to the sub-questions to see if they would be able to tell whether the steps carried out in solving such a problem are understood in the context of limits or not.

Methodological approach

In developing categories that would guide our analysis we devised the table adapted from the work of Gray and Tall (2001) in Figure 1 which shows *a spectrum of performance* that reflect the sophistication of thinking of the student (the spectrum starts with pre-procedure, procedure, then processes and ultimately the proceptual conception).

In our study, we noted that there were students who attempted some questions but used the incorrect procedures or processes to arrive at an incorrect answer. In addition, in some cases where the symbolism was observed as proceptual, the reasoning behind such an answer was faulty. These three categories are not included in the procept framework. So, in our study, we have added three extra categories (levels 1, 3, and 5) to the procept framework (as shown in Table 1) to accommodate these missing elements:

Table 2. Categories, descriptions and indicators for data analysis

LEVELS/CATEGORIES	DEFINITION/DESCRIPTION	RECOGNITION RULE/INDICATOR
Level 0: Pre-procedure	When no solution or partial solution is provided for a question.	This is a situation where the student leaves the question blank without providing a solution or where a student does not complete the response to the question
Level 1: Erroneous procedure	When the solution provided by the student is incorrect due to failure to execute the procedure accurately.	This is a situation where the procedure followed by the student is not accurate or correct.
Level 2: Procedure	The particular method(s) (sequence of steps) used accurately by an individual at a given time to solve a particular mathematics question	Sequential steps followed being correct or accurate.
Level 3: Erroneous process	The incorrect assumption that a question is a particular mathematical process	When a student thinks that a question evokes a particular mathematical process, albeit, erroneously. This is the stage whereby a student will implement the procedures at different stages of the process but does not master the input and output conception of a process.
Level 4: Process	When students are able to solve a mathematics task in a variety of ways but focusing on the input and the output.	Did the mathematics flexibly and efficiently without errors. The focus here is on the input and the output rather than the

		individual steps or procedures along the way.
Level 5: Erroneous procept	When dualism of symbolism is observed without proper understanding.	When dualism of symbolism is observed but incorrect explanation or working is displayed.
Level 6: Procept	When dualism of symbolism is observed.	Observing the duality of mathematical symbols as both a process and a concept/object.

RESULTS

The presentation of the results follows the order in which the research questions appear in the paper. The research questions read: (1) How do Social Science University students understand mathematical symbols representing the idea of limit? (2) What methods or techniques do they employ in solving mathematical tasks on limits? And (3) What justification(s) (if any) do students give for a preferred method in solving tasks over the other? The first research question is answered in two tasks (Tasks 1 and 3) that represented the idea of limit using symbols. As highlighted earlier, the two tasks were analysed using different parts of the theoretical framework, hence their results will be presented in two parts, (a) and (b).

(a) **Students' understood the mathematical symbols representing the idea of limit as non-proceptual, erroneous proceptual or proceptual**

When asked in Task 1 if the expressions $f(x) \to L$ as $x \to a$ and $\lim_{x \to a} f(x)$ mean one and the same thing, 28 (≈46.7%) students said no, while 32 (≈53%) students said yes. Students who said no were classified as not having a proceptual conception of the symbolism used. Of the 32 students who said yes, 7 (≈11.7%) of them backed up their choice of answer with faulty explanations. These students who said yes with faulty reasoning were classified as having an erroneous proceptual conception while those who gave mathematically correct reasoning were classified as having proceptual conception of the symbolism. Some of the responses belonging to the mentioned categories follow:

Non-proceptual

Student 31 below is representative of students who said 'no' in Task 1:

S31: They do not mean one and the same thing because if they were the same they would both be limits but $f(x) \to L$ as $x \to a$ is not a limit. $f(x) \to L$ as $x \to a$ means that $f(x)$ is approaching a limit as $x \to a$ while $\lim_{x \to a} f(x)$ is a limit function itself.

S31 explicitly says that $f(x) \to L$ as $x \to a$ is not a limit. This means that he does not perceive the symbolism as representing the limit as the concept but a process since he says that in this case the function is approaching the limit as $x \to a$. He refers to the symbolism $\lim_{x \to a} f(x)$ as the limit function and not as the limit. The two symbolisms are classified as the process and the concept respectively, although with an inaccurate technical terms. As highlighted earlier, it could be because the first has the dynamic feel while the second has the static one. Hence, they are not observed both as procepts.

Erroneous proceptual

An example of erroneous proceptual explanation is provided by student 22:

S22: $f(x) \to L$ as $x \to a$ mean one and the same thing because letters are being used interchangeably. So $f(x) \to L$ can be written as a form $x \to L$ and $x \to a$ in the form $f(x) \to a$.

In the erroneous proceptual conception, the use of symbols seems to be a major problem in terms of interpretation. The symbols are just used in such a way that they do not make sense at all (S22). This may be a generalisation about the use of symbols emanating from some mathematical contexts where symbols are used arbitrarily. For example, in calculus when writing functions, letters used are not restricted to only one variable; we may have $f(x), f(t), f(w)$, etc. depending on the relevance of the variable. either in the expression or the equation.

Proceptual

Finally, students 44 can be said to be on the proceptual level based on the response provided which is reminiscent of students in this category:

S44: They do mean one and the same thing but written in a different manner, that is, $\lim_{x \to a} f(x)$ is the limit of $f(x), L$, as x approaches a.

S3: Yes! Because the limit of $f(x)$ as x approaches a is the same as saying $f(x)$ approaches L when x approaches a.

The symbols used are said to mean one and the same thing. This means that each of them is seen both as the process of approaching and the concept of limit. Thus the students in this category have the procept conception of the limit symbols used.

(b) **Students' understood the infinite limits with constants as either equal to or not equal to the one without constants based on procedure**

When dealing with limits at infinity, as highlighted earlier, part of the procedure is to divide every term (in both the numerator and the denominator) by the highest power of the variable that occurs in the denominator, in this particular case x is such a variable, and every term that will end up with the structure $\frac{a}{x^n}$, (where a is a constant and $n > 0$) will tend to zero when x approaches infinity. Such terms can therefore be left out because zero is the identity element for addition and the expressions remain equal with the initial one without such terms. Some students may even go further to check the value of the limit whereas others may simply go as far as the stated steps. Since the question needed mathematically sound reasoning, some students approached the problem from different perspectives.

Task 3 read: Is $\lim_{x\to\infty}\frac{8x^3+5}{2x^2+1} = \lim_{x\to\infty}\frac{8x^3}{2x^2}$?

(i) If yes, why do you think so?

(ii) If no, why do you think so?

For this task 46 students saw the two expressions as equal while 13 said that they are not (one student (S40) did not respond to the question). Four categories of responses were generated from the 39 out of 46 students and two categories were from 6 students out of 13 students, the other *7* gave individual responses. Of the 46 students, **7** of them gave individual responses. The reasoning given in most cases for agreeing to the equality of the two limit expressions were not mathematically sound or correct. Students who denied the equality of the two limits were already incorrect. Some responses demonstrating existence of these categories follow:

Yes, they are equal

The equality of the two expressions were based on the understanding that (i) limits of constants are zeros, (ii) adding a very small number to a large number has no effect or impact (iii) that the simplification of the two expression in dividing by a variable with a higher degree in the denominator leads to the left hand side function being equal to the right hand side one and (iv) that they have the same limit. Excerpts showing these erroneous procedure conception of understanding now follow:

(i) 5 and 1 are constants and their limit is zero [10 students]

S51: Yes they are the same because a limit of a constant is zero. 5 and 1 are constants therefore there is no limit.

This reasoning is not in line with how the limit value is attained or in comparing the two expressions. Thus the procedure for computing limits of this nature is not fully grasped, hence erroneous.

(ii) 5 and 1 have no impact [13 students]

S59: Yes, because as x approaches infinity the limit of the function won't be much affected by the constants 5 and 1, in the numerator and denominator respectively.

This is a rational function whose result depends mostly on the nature of the denominator. If it were a polynomial expression ignoring the 5 or 1 for x values tending to infinity would be reasonable in terms of their impact when added to large numbers but with regard to a rational function what is important is realising that when simplification is done the process of $f(x)$ tending to L, results in zero for those terms. This is the erroneous procedure level as the exclusion of 5 and 1 is based on the fact that when dividing by the highest power of the variable in the denominator the constant terms do not affect the results without explaining why this is the case. It is as if the limits of the constants are considered instead of the limits of their quotients when division by the variable with the highest power is performed on them when coordinated with the limit operator.

(iii) *They become equal after simplification* [9 students]

S42: Yes, because $\lim_{x\to\infty}\frac{8x^3+5}{2x^2+1}=\lim_{x\to\infty}\frac{\frac{8x^3}{x^2}+\frac{5}{x^2}}{\frac{2x^2}{x^2}+\frac{1}{x^2}}=\lim_{x\to\infty}\frac{\frac{8x^3}{x^2}+0}{\frac{2x^2}{x^2}+0}=\lim_{x\to\infty}\frac{8x^3}{2x^2}$.

S31: Yes, because according to the properties of limits while solving limits which are at infinity, we only use the variables with the highest power, and we exclude the constants.

When looking at the verbal response by S42 we could have concluded that the student is at the process level of understanding, but the working shows some flaws. The process of tending to the limit value is applied to some terms but not the others while the limit operator is still carried through. This is incorrect as the coordination of $f(x)\to L$ and $x\to a$ has to be performed simultaneously. S31's generalisation of excluding constants is also not backed up with any mathematically valid reasons. The realisation of their limits being zero when solving the tasks was a necessary justification in this case to show that the student understands the procedure of computing limits at infinity.

(iv) *They approach the same number* ∞ or approach ∞ [8 students]

S32: This is because they approach the same number

S9: Yes, because their answers both are approaching infinity.

S10: Yes. Both limits are approaching infinity and the highest degree of both functions are in the numerator that means both answers will be infinity.

S32 is the only student who referred to infinity (∞) as being a specific number while it is a symbol that is used when numbers outgrow the finite bounds. Getting the same limit value for any two expressions cannot be concluded to their equality (S9). This is because these are not the only two expressions in limits that result in ∞ as the answer (which shows nonexistence of the limit). To say that the limit approaches infinity (S10) is different from saying that the limit is infinity (∞). It is equally incorrect to say both their answers (S9 and S10) approach infinity without specifying that it is $f(x)$ that tends to infinity (∞). Thus, in this case also the procedure of obtaining limits at infinity has been erroneous.

No, they are not equal

Denial of equality of the two limits on the understanding that (i) dividing by the variable with the highest degree in the denominator produced different expressions for limits and (ii) that the expressions had different limits values.

(i) *Division by the highest power of the variable in the denominator* (x^2) *gives different results* [4 students]

S4: No, it is because when finding the limit of a number approaching an infinity, we divide by the highest power of the denominator and as we do so the results become $lim\frac{8x}{2}$ which is different from $\lim\frac{8x^3}{2x^2}$.

S16: The function because as we simplify the function it gives the different answer from the answer already given, $\lim_{x\to\infty}\frac{8x^3+5}{2x^2+1}=4x$ and not $\lim_{x\to\infty}\frac{8x^3}{2x^2}$.

In both cases the students are let down by failure to judge the equality of the resulting expressions. The students take $4x$ as the limit value which shows that they did not apply the limit operator on it to get ∞ as the answer. Thus, the procedure was erroneous.

(ii) *Different limits are obtained* [2 students]

S6: No, they are way too different because $\lim_{x\to\infty}\frac{8x^3+5}{2x^2+1}$ is 4 while $\lim_{x\to\infty}\frac{8x^3}{2x^2}$ is 0. So the two operations or equations are way too different. The other one approaches 0 while the other one approaches 4.

S58: No, because it doesn't make mathematical sense and $\lim_{x\to\infty}\frac{8x^3+5}{2x^2+1}$ is not equal to $\lim_{x\to\infty}\frac{8x^3}{2x^2}$ because when worked out they both give different result.

The denial for equality is already not a correct response. The first category of the no response has been separated from the second because in the first, students have clearly shown or said that they divided by the highest power of the variable in the denominator while this was not the case with the second category.

The students who said yes (the correct response) and the correct reasoning for this task would have been said to have reached the process and the concept level of the framework of analysis (performance outcome). This is because whether the question is responded to verbally or by a combination of carrying out the procedure(s) algebraically the student would still need to focus on the flexibility of getting the input and output by alternative means (Gray & Tall, 1994). Attaining the output requires both the knowledge of steps to be taken and the implementation of the coordination of the processes, $f(x)\to L$ and $x\to a$ in moving from the left-hand side to the right-hand side. While this is what we thought was the case, the reasoning did not match our judgement based on the yes response. Only one student, S40, who did not give any response was at the pre-procedure level. Students in the study of Maharaj (2016) also had problems with the computation of limits at infinity.

Methods used by students in solving tasks covered the adapted levels of sophistication of performance (from pre-procedural to process level)

The choice of how to solve a task depends very much on how the student understands the task the way it is presented (research question 2). The first part of Task 2 is presented. The sub-questions are discussed in the next subsection as their data gave answers to the third research question.

Task 2 (First part) read: Find $\lim_{x\to 3}\frac{x^3-27}{x-3}$.

In responding to this task, the two methods that were used are the numerical (table) and the algebraic. Each of these consists of a set of procedures to be followed and processes leading to the end result, the limit value which is understood to be an object/concept. Of the 60 students, 27 used the numerical (table method) while 33 used the algebraic method. Three students were at the pre-procedure stage because they did not complete their work.

Erroneous procedure level was achieved by 35 students. Twenty (20) students reached the procept stage. Students who reached the procept stage mastered both the procedures and processes that resulted in the correct output. Erroneous procept stage was achieved by 2 students who committed an error at when deducing the limit value at the procedural stage. All levels of outcome of the framework of analysis were realised. The excerpts that follow show these levels in students' workings (excerpt the pre-procedure level where the question was not attempted).

Erroneous procedure

S13: $\lim_{x \to 3} \frac{x^3-27}{x-3} = \frac{x\,x\,x-27}{x-3} = \frac{3^2-27}{-3} = \frac{9-27}{-3} = \frac{-18}{-3} = 6$

The factoring of *x* was incorrect as it was not a common factor. The limit operator was also left out in the second step while *x* still existed as a variable.

Erroneous procedure and process

S41: x	$f(x)$	x	$f(x)$
2.9	26.11	3.01	27.0901
2.99	26.9101	3.001	27.009001
2.999	26.991001	3.0001	27.00090001

It approaches 27

S41 did not clarify as to what approaches 27. It only became clear in his reasoning (presented in the next subsection) that 27 is the value that he thinks 3 approaches instead of the value being approached by $f(x)$. Thus the coordination of the processes, $f(x) \to L$ as $x \to a$, was faulty, hence the erroneous procedure and process as the output was not arrived at by proper reasoning. He realised that the substitution method did not work as the calculations produce 0 divided by 0.

Procedure and process

S55: x	$f(x)$	x	$f(x)$
2.9	26.11	3.01	27.0901
2.99	26.9101	3.001	27.009001
2.999	26.991001	3.0001	27.00090001

27 Answer

S55 got all the steps correct (procedure within the process and the process in terms of input and output) that let to finding the correct limit value.

OR

S19: $\lim_{x \to 3} \frac{x^3-27}{x-3} = \lim_{x \to 3} \frac{(x-3)(x^2+3x+9)}{x-3}, x \neq 3 = \lim_{x \to 3} x^2 + 3x + 9$ = 27

$$(a-b)(a^2+ab+b^2)$$

$$(x-3)(x^2+3x+3^2)$$

All the steps taken to getting the limit value by S19 are correct. Thus, he qualified to be classified under the process level in terms of input and output

The choice of methods for solving tasks was based on appropriateness, accuracy or efficiency

In solving the tasks on limits there are some rules or procedures that one must follow depending on the nature of the task. It is not every method that can be applicable to all situations. Such a choice requires some understanding of why one method is more appropriate to use than the other known methods to a given situation. In this study, the students explained their choice of methods according to suitability or appropriateness to the task or their efficiency. Supplementary questions to the choice of method were posed in Task 2. This part is an attempt to answer research question 3.

Twenty-seven students who used the numerical (table method) said that this method is reliable while 32 out of 33 students who used the algebraic method said to be easier and it saves time. One student (S24) did not respond to these sub-questions. The students said that the numerical method is reliable because it allowed them to avoid getting 0/0 through substitution of 3 in the place of x in the function. Those who used the algebraic method argued that the method saves time as compared to the numerical method which they acknowledged to be the other method that they know. The excerpts of responses showing all the levels (pre-procedure to process in terms of input and output) of sophistication of the students' performance now follow.

Pre-procedure

S34:

(i) In substituting 3 for x the values would be divided by a 0. So if I don't then the $x-3$ are to move one another with the one on top making $x \neq 3$.

(ii) No

(iii) If I use the power rule, powers will be long since we have a numerator and the denominator $\frac{d}{dx}$may be hard to find.

The student was aware that substituting 3 for x would produce division by zero and did not know what to do next. Finding the limit value was however confused with finding the derivative by making reference to the power rule. Because of the given explanation, the student did not solve the given task.

Erroneous procedure

S13:

(i) What kind of method should I use? Should I substitute or make a table?

(ii) Table method

(iii) I chose substitution because I did not know how I was going to make a table even though my substitution gave me a problem, but I factored out xs ten divide there after I substituted with the 3.

The student is aware that she did not master the chosen method (procedure) but had problems with the table method.

Erroneous Process

S41

(i) I saw that when I substituted it came to a point where I get 0 and 0 divided by 0 is undefined. I saw that when I use the table, it gives me the values which are very close to each other.

(ii) Substitution method is the other method that I know.

(iii) I chose table because it gives me the exact value which 3 approaches.

S41 realised that the substitution method did not work as the calculations produced 0 divided by 0. He chose the table method because it gave him the value which 3 approaches. The reasoning is faulty as shown earlier. It is the value that the functional values are approaching that is the limit value. So, the input and output (process) connection is faulty.

Process

S55:

(i) I thought of substituting x with 3 but the answer I got was 0/0. I decided to use the table method.

(ii) Yes using algebra.

(iii) I used the table method because I forgot to break $x^3 - 27$ algebraically. So, I thought of the table as the next one.

S55 realised that the substitution of 3 posed some problems. Using algebra was also problematic for him by not knowing how to factor the difference of two cubes. We assume that this is what he means by breaking $x^3 - 27$ algebraically.

AND

S19:

(i) I thought of using the table then I realised that it takes long. Then I thought of factoring $x^3 - 27$ so $(x - 3)$ will be cancelled.

(ii) Yes, the table method.

(iii) It was easier and it saves time.

S19 seems to know the two methods that were applicable in this case, table and algebraic. He however chose to use the algebraic method on the basis that the table method would take him a longer time. Thus, the method (procedure) was chosen based on its efficiency.

DISCUSSION

For the first Research Question about how students perceive the symbols representing limits, we have found out that the symbols were perceived as non-proceptual, erroneous proceptual or proceptual. We believe that the students displayed such conceptions because in teaching such terminology is not made explicit but implicit which exhibits the students to truly conceptualise the symbols the way the framework suggests, proceptual understanding. The findings do overlap with the findings of Güçler (2014) which addressed the question: "How do one instructor and his students use and think about the limit notation in a beginning-level undergraduate calculus classroom?" (p.251). The findings of the study show that students perceived the limit as a process when using its notation, which is just part of the spectrum of outcomes of the Procept Theory.

For the second Research Question on how the students solved the problems on limits, our findings show that all the levels of sophistication of the adapted Procept Theory were displayed in the students' work, from pre-procedural to Proceptual level. The most difficult tasks on limits to solve were those that involved limit at infinity. Infinity as a notation, ∞, was problematic for students in the studies of both Maharaj (2010) and Jones (2015). We believe that this symbol which is used to show when numbers outgrow the finite bounds is very difficult for students to handle as it is something that cannot be shown on the real number line because of its metaphysical nature. Students also had problems with factoring. In this study, the factoring was that of difference of cubes while in the study of Maharaj (2010), the factoring involved the difference of squares involving radicals. As in the study of Denbel (2014), some students in the reported study thought only about the manipulative aspects and did focus on the idea of the limit. Some similarities also exist with the findings of Moru (2009) in that at the erroneous procedural level some students denied the existence of the limit where the function was not defined. This was concluded from getting 0/0 after substituting $x = a$ in the function. This was the same finding in the study of Denbel (2014). In both studies there are students who said that the limit is not being defined at the point, $x = a$, when it is actually the function that is not defined at that point

The students' choice of methods (Research Question 3) was based on appropriateness, accuracy or efficiency in terms of time and not necessarily in the way they perceived the symbolism from the framework of the Procept Theory as it was not part of their vocabulary. This part of allowing students to explain their answers was unique to this study, hence there is not comparison that can be made with other studies elsewhere.

This research has shown that as we teach limits we should explicitly use the language of the framework that is relevant to the idea being taught, in this case, the Procept Theory. On the other hand it could be argued that since this is not the only relevant theory, it could inhibit the versatility of students' thinking. Moreover the students may not easily get accustomed to this complex terminology which may only have meaning to the mathematics educators as it falls within their field of study. This view is supported by the study of Güçler (2014) on how students think about the limit notation. The findings show that although the instructor explicitly differentiated between the process and the product aspects of the limit, students still perceived the limit as a process when using its notation.

Although the procept theory has been used as the main framework for data analysis, additions that we made in the framework seem to have allowed us to classify some responses which did not exactly fit into the original framework. This is one of the major contributions that this study has made. Another important aspect of the idea has been that of having access to students' thinking with regard to their choice of methods when solving the tasks on limits. The type of questions that the students asked themselves when responding to the task have definitely been useful in understanding why students responded to the tasks the way they did. It seems to have been important not only to the researchers but also to students who had to reflect on what they were doing so that they can learn to make choices consciously.

CONCLUSION

The findings of the study have shown that the Social Science students displayed all the conceptions described in the adapted Procept Theory, pre-procedural to proceptual. Students encountered some problems in solving problems on limits. The most difficult procedure for students was that of factoring difference of squares. Handling limits at infinity was also problematic as the symbol ∞ seemed to be too abstract for some. Seeing the symbolism of limits as both the process and the product was conceptualised by very few students. In some cases the errors were displayed in the language that was used to explain the answers. The errors showed that students did not only have problems in explaining the technical terms but they also had problems in naming the technical terms. This was indicative in the manner in which the students were supporting their choice of methods or in explaining their answers. We would suggest that conducting more studies that focus on the language of limits, which is not necessarily confined to algebraic symbolism would be of great help to understanding students' conception of limits. Thus it could be a step in the right direction on how limits together with the associated language and symbolism can be used in teaching.

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