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THE BINARY GOLDBACH CONJECTURE

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ABSTRACT

In 1742 German mathematician Christian Goldbach, in a letter addressed to Leonhard Euler, proposed a conjecture. Today the Goldbach Conjecture is one of the oldest problems in mathematics, it has fascinated and inspired many mathematicians for ages. The modern day version of the Binary/Strong Goldbach conjecture asserts that:

Every even integer greater than 2 can be written as the sum of two primes.

The conjecture had been verified empirically up to $4*10^{18}$, its proof however remains elusive, which seems to confirm that:

Some problems in mathematics remain buried deep in the catacombs of slow progress ... mind stretching mysteries await to be discovered beyond the boundaries of former thought.

Avery Carr (2013)

The research was aimed at exposition, of the intricate structure of the fabric of the Goldbach Conjecture problem. The research methodology explores a number of topics, before the definite proof of the Goldbach Conjecture can be presented. The Ternary Goldbach Conjecture Corollary follows the proof of the Binary Goldbach Conjecture as well as the representation of even numbers by the difference of two primes Corollary. The research demonstrates that the Goldbach Conjecture is a genuine arithmetical question.

Keywords: Goldbach conjecture, Binary Goldbach conjecture, Ternary Goldbach conjecture, sum of primes, primes in arithmetic progression, prime number theorem

LITERATURE REVIEW

Christian Goldbach in his letter to Leonhard Euler dated 7 June 1742, stated that every even number greater than 2 can be written as a sum of two prime numbers:

$$(1.1) \quad 2m = p_i + p_j \quad \forall m \in \mathbb{N} \mid m \geq 2, \text{ and prime numbers } p_i, p_j \in \mathbb{N} \mid p_i \leq p_j$$

At the Second International Congress of Mathematicians in Paris, in August 1900, David Hilbert proposed a list of 23 mathematical problems, which he defined as problems of "deep significance for advancement of the mathematical science".

A great problem must be clear, because, what is clear and easily comprehended attracts, the complicated repels us ... It should be difficult in order to entice us, yet not completely inaccessible, lest it mock our efforts.

David Hilbert

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Hilbert placed the Goldbach Conjecture, together with the Riemann Hypothesis and the Twin Primes Conjecture, as the 8-th problem on the list. The Goldbach Conjecture itself consists of two parts, the Binary (or Strong) Goldbach Conjecture and the Ternary (or Weak) Goldbach Conjecture. Although a massive effort has been exerted, yet both of them remain unproven. Many mathematicians worked on this problem, notably Brun (1919), Hardy & Littlewood (1923), Vinogradov (1934), Chen (2002), Kaniecki (1995), Deshouillers et al (1997) to name a few. Many more of the distinguished works are presented and/or referred to by Wang (2002).

The Goldbach conjecture has even been a theme of a storybook written by Apostolos Doxiadis (2012), under the title "Uncle Petros and Goldbach's Conjecture". The book tells the story of a man who dedicated his life to the research of the Goldbach Conjecture, searching for its proof. The publisher even offered a prize of 1 million dollars for a proof, as a part of a publicity stunt.

2. The Binomial Expansion $2^{\left(n+\mathcal{G}_{(n)}\right)}$

Within the scope of the paper, prime gap of the size g $\in \mathbb{N} \mid$ g ≥ 2 is defined as an interval between two primes p_i, $p_{(i+1)}$, containing (g - 1) composite integers. Maximal prime gap of the size g, is a gap strictly exceeding in size any preceding gap. All calculations and graphing were carried out with the aid of $Mathematica^{®}$ software.

One of the step-stones in this paper, is the bound on the maximal prime gaps. This part presented here, is an extract from a paper by Feliksiak [9]. For all n $\in \mathbb{N} \mid$ n ≥ 5, we make the following definitions:

Definition 2.1 (Scaling factor). $\xi = \frac{\log_{10}(\frac{n}{24})}{\log_{10}(24)}$

Definition 2.2 (Interval length). $c = \mathcal{G}_{(n)} = \left\lfloor 5\left(\log_{10}\ n\right)^2 \right\rfloor$

Definition 2.3 (Interval endpoint). $t = (n + c)$

Definition 2.4 (Binomial coefficient).

$$\mathcal{M}_{(t)} = \binom{n+c}{n} = \left(\frac{(n+c)!}{(n!\ c!)}\right)$$

Definition 2.5 (Logarithm of the binomial coefficient).

$$\log \mathcal{M}_{(t)} = \log\left(\frac{(n+c)!}{(n! \times c!)}\right) = \log(t!) - \log(n!) - \log(c!) = \sum_{k=1}^{c} \log(n+k) - \sum_{k=1}^{c} \log k$$

Lemma 2.6 (Upper and Lower bounds on the log of *n*!).

The bounds on the logarithm of n! are given by:

$$(2.1) \qquad n\log(n) - n + 1 \leq \log(n!) \leq (n+1)\log(n+1) - n \quad \forall n \in \mathbb{N} \mid n \geq 5$$

Proof.

Evidently,

$$\log(n!) = \sum_{k=1}^{n} log(k) \quad \forall n \in \mathbb{N} \mid n \geq 2 \tag{2.2}$$

Now, The pertinent integrals to consider are:

$$\int_1^n \log(x)\ dx \leq \log(n!) \leq \int_0^n \log(x+1)\ dx \quad \forall n \in \mathbb{N} \mid n \geq 5 \tag{2.3}$$

Accordingly, evaluating those integrals we obtain:

$$n\log(n) - n + 1 \leq \log(n!) \leq n\log\left(\frac{(n+1)}{e}\right) + \log\left(\frac{(n+1)}{e}\right) + 1 = (n+1)\log(n+1) - n \tag{2.4}$$

Concluding the proof of Lemma 2.6

Remark 2.1.

Observe that log $M_{(t)}$ is a difference of logarithms of factorial terms:

$$\log \mathcal{M}_{(t)} = (\log(t!) - \log(n!) - \log(c!))$$

Consequently, implementing the lower/upper bounds on the logarithm of n! for the bounds on log $M_{(t)}$, results in bounds of the form:

$$\log\left(\frac{(t+k)^{(t+k)}}{(n+k)^{(n+k)}\ (c+k)^{(c+k)}}\right) \quad \text{for } \forall\ k \in \mathbb{N} \cup \{0\} \tag{2.5}$$

Keeping the values of *c, n* and *t* constant and letting the variable *k* to increase unboundedly, results in an unbounded monotonically decreasing function. When implementing the lower/upper bounds on the logarithm of *n*! for the Supremum/Infimum bounds on log $M_{(t)}$, the variable *k* appears only with values *k* = {0, 1} respectively. The combined effect of the difference of the logarithms of factorial terms in log $M_{(t)}$ and the decreasing property of the function 2.5, imposes a reciprocal interchange of the bounds 2.1, when implementing them for the bounds on log $M_{(t)}$.

Lemma 2.7 (log $M_{(t)}$ Supremum Bound)**.**

The Supremum Bound on the logarithm of the binomial coefficient $M_{(t)}$ is given by:

$$\log \mathcal{M}_{(t)} \leq \log\left(\frac{t^t}{n^n\ c^c}\right) - 1 = \mathcal{UB}_{(t)} \quad \forall n \in \mathbb{N} \mid n \geq 5 \tag{2.6}$$

Proof.

Evidently, by Lemma 2.6 we have:

$$(n\log(n) - n + 1) \leq \log(n!) \tag{2.7}$$

Substituting from the inequality 2.7 into the Definition 2.5 we obtain:

$$\begin{aligned}(2.8)\quad & (\log(t!) - \log(n!) - \log(c!)) \\ & \leq ((t\log(t) - t + 1) - (n\log(n) - n + 1) - (c\log(c) - c + 1)) \\ & = t\log(t) - n\log(n) - c\log(c) - 1 = \log\left(\frac{t^t}{n^n\ c^c}\right) - 1\end{aligned}$$

Consequently,

$$\log \mathcal{M}_{(t)} \leq \log \left(\frac{t^t}{n^n\ c^c} \right) - 1 = \mathcal{UB}_{(t)} \tag{2.9}$$

The Supremum bound $UB_{(t)}$ produces an increasing, strictly monotone sequence in $\mathbb{R}$.
At $n = 5$, the difference $UB_{(t)}$ - log $M_{(t)}$ attains 0.143365 and diverges as n tends to infinity. Therefore, Lemma 2.7 holds as specified.

Lemma 2.8 (log $M_{(t)}$ Infimum bound)**.**
The Infimum Bound on the natural logarithm of the binomial coefficient $M_{(t)}$,
for all $n \in \mathbb{N} \mid n \geq 5$ *is given by:*

$$\log \mathcal{M}_{(t)} \geq \log \left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)}\ (c+1)^{(c+1)}} \right) = \mathcal{LB}_{(t)} \tag{2.10}$$

Proof.
From Lemma 2.6 we have:

$$\log (n!) \leq n \log (n+1) - n + \log (n+1) \tag{2.11}$$

Substituting from the inequality 2.11 into the Definition 2.5, we obtain:

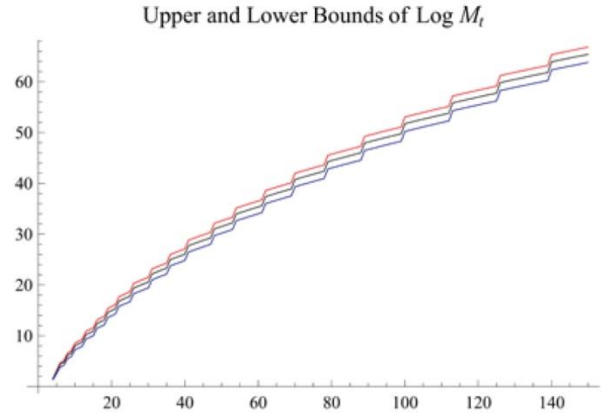
$$\begin{aligned} &(2.12) \quad (\log (t!) - \log (n!) - \log (c!)) \\ &\geq t \log (t+1) - n \log (n+1) - c \log (c+1) + \log (t+1) - \log (n+1) - \log (c+1) \\ &= \log \left(\frac{(t+1)^t}{(n+1)^n\ (c+1)^c} \right) + \log \left(\frac{(t+1)}{(n+1)\ (c+1)} \right) \end{aligned}$$

Consequently,

$$\log \mathcal{M}_{(t)} \geq \log \left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)}\ (c+1)^{(c+1)}} \right) = \mathcal{LB}_{(t)} \tag{2.13}$$

The Infimum bound $LB_{(t)}$ produces an increasing, strictly monotone sequence in $\mathbb{R}$. At $n = 5$, the difference log $M_{(t)}$ - $LB_{(t)}$ attains 0.455384 and diverges as n tends to infinity. Therefore, Lemma 2.8 holds as specified.

3. Maximal Prime Gaps



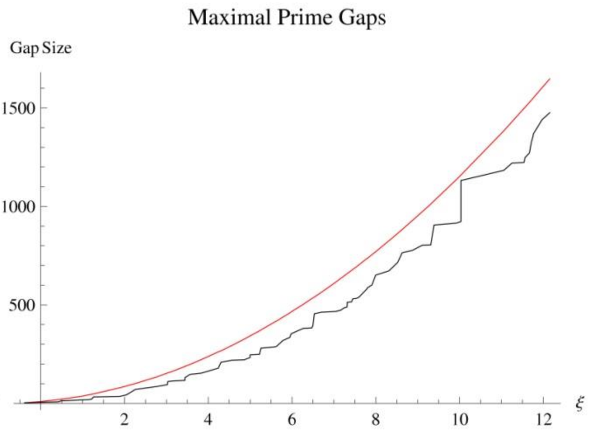


Figure 1. The left drawing shows the graphs of the lower (blue) and upper (red) bounds versus log $M_{(t)}$ (black). The right drawing shows the graph of $G_{(n)}$ (red) and the actual maximal prime gaps (black) with respect to ξ as given by the Definition 2.1. The graph has been produced on the basis of data obtained from C. Caldwell as well as from T. Nicely tables of maximal prime gaps.

We begin with a preliminary derivation. Since the integers from 1 to n contain $\left\lfloor \frac{n}{p} \right\rfloor$ multiples of the prime number p, $\left\lfloor \frac{n}{p^2} \right\rfloor$ multiples of p^2 etc. Thus it follows that:

$$n! = \prod_p p^{u_{(n,p)}}; \ where \ u_{(n,p)} = \sum_{m \geq 1} \left\lfloor \frac{n}{p^m} \right\rfloor$$

In accordance with the Definitions 2.2, 2.3 and 2.4 we obtain:

$$\mathcal{M}_{(t)} = \prod_{p \leq t} p^{\mathcal{K}_p}$$

where

$$\mathcal{K}_p = \sum_{m=1}^{\infty} \left(\left\lfloor \frac{t}{p^m} \right\rfloor - \left\lfloor \frac{n}{p^m} \right\rfloor - \left\lfloor \frac{\mathcal{G}_{(n)}}{p^m} \right\rfloor \right)$$

it follows that

$$\mathcal{K}_p \leq \left\lfloor \frac{\log t}{\log p} \right\rfloor$$

and so by the above, Lemma 2.7 and 2.8 we have:

$$(3.1) \quad \mathcal{LB}_{(t)} \leq \log \mathcal{M}_{(t)} = \log \prod_{p \leq t} p^{\mathcal{K}_p} = \sum_{p \leq t} \mathcal{K}_p \log p \ \leq \ \mathcal{UB}_{(t)} \quad \forall n \in \mathbb{N} \mid n \geq 5$$

Where p is as usual a prime number.

Lemma 3.1 (Prime Factors of $M_{(t)}$).

The case when there does not exist any prime factor p of $M_{(t)}$ in the interval from n to t, $\forall n \in \mathbb{N} \mid n \geq 8$*, imposes an upper limit on all prime factors p of $M_{(t)}$. Consequently in this particular case, every prime factor p must be less than or equal to s* = ⌊ *t*/2⌋.

Proof.

Let p be a prime factor of $M_{(t)}$ so that $\mathcal{K}_p \geq 1$ and suppose that every prime factor $p \leq n$. If

$$s < p \leq n$$

then,

$$p < (n + \mathcal{G}_{(n)}) < 2p$$

and

$$p^2 > \left(\frac{(n + \mathcal{G}_{(n)})}{2}\right)^2 > (n + \mathcal{G}_{(n)})$$

and so $\mathcal{K}_p = 0$. Therefore $p \leq s$ for every prime factor p of $\mathcal{M}_{(t)}$, for any $n \in \mathbb{N} \mid n \geq 8$. □

3.1. **Maximal prime gaps upper bound.**

The binomial coefficient $M_{(t)}$:

$$2^{t/2} < n^{\frac{c}{2}} < \exp\left(\mathcal{LB}_{(t)}\right) \leq \mathcal{M}_{(t)} = \left(\frac{(n+c)!}{(n! \times c!)}\right) \leq \exp\left(\mathcal{UB}_{(t)}\right) < n^{\frac{2c}{3}} < 2^t$$

$$\forall n \in \mathbb{N} \mid n \geq 22$$

The bounds on the logarithm of $M_{(t)}$ are given by **Lemma** 2.7 and 2.8:

$$(3.2) \quad \mathcal{LB}_{(t)} = \log\left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)} \; (c+1)^{(c+1)}}\right)$$

$$\leq \log \mathcal{M}_{(t)} = \sum_{k=1}^{c} \log(n+k) - \sum_{k=1}^{c} \log k \leq \log\left(\frac{t^t}{n^n \; c^c}\right) - 1 = \mathcal{UB}_{(t)}$$

$$\forall n \in \mathbb{N} \mid n \geq 5$$

The proof of the Maximal Gaps Theorem implements the Supremum bound function $\mathcal{UB}_{(t_s)}$ Due to the fact that the Supremum function $UB_{(t)}$ applies values of *n*, *c* and *t* directly, it imposes a technical requirement to generate a set of pertinent values, to correctly approximate the interval *s*. This is to ascertain that the generated interval is at least equal to *s*, and the corresponding value of *c* is correct. Respective definitions follow:

Definition 3.2. $n_s = \frac{n}{2}$

Definition 3.3. $c_s = 5\left(\log_{10} \; (n_s)\right)^2 + 1$

Definition 3.4. $t_s = n_s + c_s$

Theorem 3.5 (Maximal Prime Gaps Bound and Infimum for primes)**.**

For any n $\in \mathbb{N}$ |n $\geq$ 8 *there exists at least one p* $\in \mathbb{N}$ | *n < p* $\leq$ *t; where p is as usual a prime number and the maximal prime gaps upper bound G*$_{(n)}$ *is given by:*

$$(3.3) \quad \mathcal{G}_{(n)} = \left\lfloor 5 \left(\log_{10}\ n\right)^2 \right\rfloor \qquad \forall n \ \in\ \mathbb{N} \mid n \geq 8$$

$$Equivalently, p_{i+1} - p_i \ \leq\ \mathcal{G}_{(p_i)}$$

Proof.

Suppose that there is no prime within the interval from *n* to *t*. Then in accordance with the hypothesis, by Lemma 3.1 we have that, every prime factor *p* of *M*$_{(t)}$ must be less than or at most equal to *s* = ⌊*t/*2⌋. Invoking Definitions 3.2, 3.3 and 3.4, Lemma 2.7, 2.8 and the inequality 3.1 we derive for all *n* $\in \mathbb{N}$ |n $\geq$ 8:

$$(3.4) \quad \mathcal{LB}_{(t)} = \log\left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)}\ \ (c+1)^{(c+1)}}\right)$$

$$\leq \log \mathcal{M}_{(t)} = \log \prod_{p \leq t_{(s)}} p^{\mathcal{K}_p} = \sum_{p \leq t_{(s)}} \mathcal{K}_p \log p \ \leq\ \log\left(\frac{(t_s)^{t_s}}{(n_s)^{n_s}\ (c_s)^{c_s}}\right) - 1 = \mathcal{UB}_{(t_s)}$$

In accordance with the hypothesis therefore, it must be true that:

$$(3.5) \qquad a_c = \log\left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)}\ \ (c+1)^{(c+1)}}\right) - \log\left(\frac{(t_s)^{t_s}}{(n_s)^{n_s}\ (c_s)^{c_s}}\right) + 1 < 0$$

Now, we apply the Cauchy's Root Test for n $\geq$ 43:

$$(3.6) \qquad \lim_{c \to \infty} \sqrt[c]{|a_c|} = \lim_{c \to \infty} \sqrt[c]{\left|\mathcal{LB}_{(t)} - \mathcal{UB}_{(t_s)}\right|} \to 1$$

At n = 43 the Cauchy's Root Test attains approx. 1.17851 and tends asymptotically to 1, decreasing strictly from above. Thus, by the definition of the Cauchy's Root Test, the series formed from the terms of the difference *LB*$_{(t)}$ - $\mathcal{UB}_{(t_s)}$ diverges as *c* increases unboundedly. Hence, in accordance with the hypothesis, inequality 3.5 diverges to negative infinity as *n* increases unboundedly. However, at *n* = 43 the difference 3.5 attains approx. 9.45885151 and diverges as *n* increases unboundedly. Hence, we have a contradiction to the initial hypothesis. This implies that for all *n* $\in \mathbb{N}$ |n $\geq$ 43:

$$(3.7) \qquad \mathcal{LB}_{(t)} - \mathcal{UB}_{(t_s)} \ >\ 0$$

Necessarily therefore, there must be at least one prime number within the interval *c* for all *n* $\in \mathbb{N}$ |n $\geq$ 43. Table 1 lists all values of *n* s.t. 8 $\leq$ *n* $\leq$ 47. Evidently, every possible sub-interval contains at least one prime number. Thus we deduce that Theorem 3.5 holds in this range as well. Consequently Theorem 3.5 holds as stated for all *n* $\in \mathbb{N}$ |n $\geq$ 8. Thus completing the proof.

Table 1. Low range $G_{(n)}$ vs. primes within the range

n	$G_{(n)}$	primes	n	$G_{(n)}$	primes
8	4	11	29	10	31, 37
11	5	13	31	11	37, 41
13	6	17, 19	37	12	41, 43, 47
17	7	19, 23	41	13	43, 47, 53
19	8	23	43	13	47, 53
23	9	29, 31	47	13	53, 59

4. The Binary/Strong Goldbach Conjecture - Discussion

From the pragmatic point of view, we incorporate into the research methodology of the Goldbach Conjecture, the aspect of prime numbers in **Arithmetic Progression** (**AP**). We begin with the Theorem by P.G. Lejeune-Dirichlet (1837).

Theorem 4.1 (Primes in arithmetic progression)**.**
Let d and *a be co-prime integers. Then the arithmetic progression a, a*+*d, a*+2*d, a*+3*d, ... contains infinitely many primes. Moreover, the density of the set:*

D = {*p,* s.t. p is prime, *p is congruent to a* (*mod d*)}

in the set of primes is 1/(*ϕ d) , where ϕ is the Euler totient function. Equivalently, the set D has infinite cardinality.*

For a proof, please consult pertinent literature. The natural numbers can be split into six disjoint equivalence classes, their union producing $\mathbb{N}$. Since prime numbers $p > 2$ are odd, there are only three pertinent equivalence classes to consider. Let's define the set *S*:

Definition 4.2.
S = {s | s = 6n + 3, or s = 6n + 5, or else s = 6n + 7} $\forall n \in \mathbb{N} \cup \{0\}$

Clearly, *S* contains all odd natural numbers $s \geq 3$. Since for all $n \in \mathbb{N}$ | *s* = 6*n* + 3, the element *s is* congruent to 0 (mod 3), consequently, this equivalence class produces composite numbers exclusively. The remaining two equivalence classes: 6*n* + 5 and 6*n* + 7, by Theorem 4.1 produce infinitely many primes each. Because all primes $p \in \mathbb{N}$ | $p \geq 3$ are odd, this implies that *S* contains all prime numbers $p \geq 3$. Let's therefore define an acronym to represent the two equivalence classes:

Definition 4.3 (Lower Prime Form Integer). LPF = {6n + 5| $n \in \mathbb{N} \cup \{0\}$}
Definition 4.4 (Upper Prime Form Integer). UPF = {6n + 7| $n \in \mathbb{N} \cup \{0\}$}

Clearly, the union of *LPF* and *UPF* equivalence classes, together with {2, 3}, contains all primes $p \in \mathbb{N}$ (It also contains composite numbers as well).

The concept of adding two prime numbers, in order to obtain an even number, is well known. It is however a challenging and a very intricate problem, a view also shared and verified by E. Calude in her research paper, Calude (2009).

From the Binomial Theorem, we have that the number of possible combinations of pairs of primes $2 \leq p_i, p_j \leq p_k$, to form a sum is given by the Binomial Coefficient $C((k+1), 2)$ where the order does not count and repetitions are permitted:

$$C_k = \frac{(k+1)!}{2! \times (k-1)!} = \frac{k(k+1)}{2} \tag{4.1}$$

Obviously there will be repeats, not only with the two prime summands reversed, but also numerically, with different pairs of primes generating the same even number. In addition to that there will be combinations involving the prime number 2. Now, for the prime numbers $p_i, p_j, p_k \in \mathbb{N}$, let's define the set **(assumed already sorted and any existing repeats discarded)**:

Definition 4.5. $\mathcal{E} = \{e,\ e \in \mathbb{N} \mid e \equiv 0(\text{mod } 2),\ e = p_i + p_j,\ \text{with } p_i, p_j \leq p_k\}$

The generated set E, contains all possible sums of two odd primes $p_i, p_j \leq p_k$, up to $2p_k$. This set may only include new elements when an additional prime number will become available in the range, or in other words, the range will be extended.

A natural question arises, "*will all of those generated distinct even numbers be consecutive*"*?* In general, depending on p_k and e, there will be values missing in the set E. **Let's define the function N°, containing strictly the count of distinct, consecutive even numbers, elements of the set E, strictly up to (not including), the first failed/skipped even number**:

Definition 4.6 $N^\circ = \sum_e 1 \quad \text{for } e \in \mathcal{E} \mid e = p_i + p_j,\ \text{with } p_i, p_j \leq p_k$

The function N° is weakly increasing, since it is only possible to generate new distinct, consecutive even numbers, with additional primes coming into consideration. Between the primes, the function N° exhibits a horizontal/level slope at the height of the count attained at the last failed/skipped even number. The expected exact count of distinct, consecutive even numbers in the range up to p_k, necessary to validate the Goldbach's Conjecture, clearly is:

$$N^\circ = \left\lfloor \frac{p_k}{2} \right\rfloor - 1 \qquad 4 \leq p_i + p_j < p_k \tag{4.2}$$

Since however, the greatest even integer that can be generated, by implementing primes $p_i, p_j \leq p_k$ is $2p_k$, some of the generated even numbers will clearly exceed p_k. The prime p_k itself will obviously recur in pairs with greater primes, this is however a secondary issue. **The true count of the generated distinct, consecutive even numbers** is contained within the range:

$$3 < N^\circ \leq p_k - 1; \quad \text{for } \ 4 \leq p_i + p_j \leq 2p_k \tag{4.3}$$

Where the lower limit, represents the first 3 even numbers 4, 6, 8, which cannot be generated by the sole use of the **AP** formulae, and p_k - 1 indicates, that we exclude the

$n \in \mathbb{N} \cup \{0\}$

number 2 from consideration. When implementing the **AP** formulae, the sum of two arbitrary LPF and/or UPF integers for all *m,* takes the form:

- **Case 1**: The even number *e* is a all LPF variety. The sum of two arbitrary LPF integers produces an even integer of the form:

  $$(4.4)\ \ e = (6m + 5) + (6n + 5) = 6\,(m + n + 1) + 4, \text{ producing } \{10, 16, 22, 28, \ldots\}$$

- **Case 2**: The even number *e* is a Mixed variety. The sum of an arbitrary LPF integer and an arbitrary UPF integer produces an even integer of the form:

  $$(4.5)\ \ e = (6m + 5) + (6n + 7) = 6\,(m + n + 1) + 6, \text{ producing } \{12, 18, 24, 30, \ldots\}$$

- **Case 3**: The even number *e* is all UPF variety. The sum of two arbitrary UPF integers produces an even integer of the form:

  $$(4.6)\ \ e = (6m + 7) + (6n + 7) = 6\,(m + n + 1) + 8, \text{ producing } \{14, 20, 26, 32, \ldots\}$$

In the formulae 4.5 and 4.6 above, we could have inserted 0 and 2 respectively. However, by inserting 6 and 8 instead, we actually carry a bit of information that will be of assistance later. **The exception here is, the case of the set of summands** {3*,* (*e* - 3)}**. This particular set of summands can only be assigned to Case 1 or Case 3. An instance which depends on the variety of *e*.**

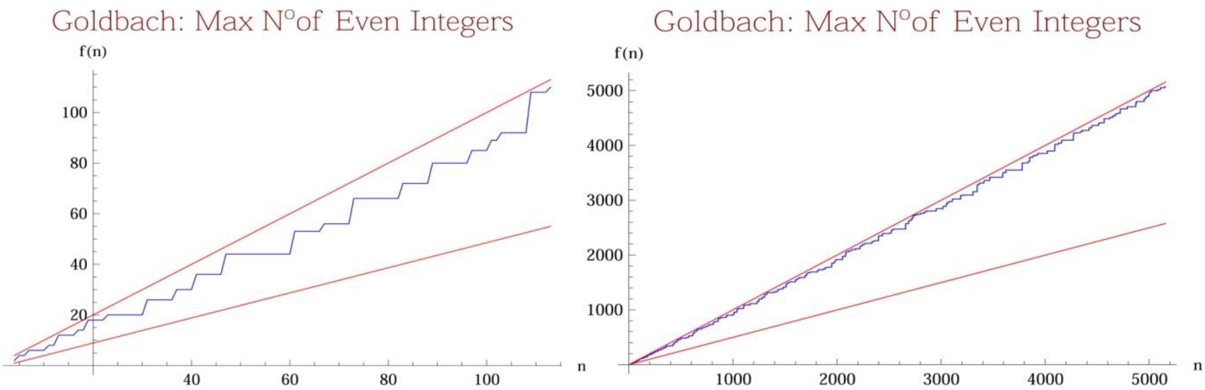


Figure 2. The drawing shows the graph of the function N° (Blue) displaying the maximal volume of distinct, consecutive even numbers generated in the range $4 \le e$ congruent to $0 (\text{mod } 2) \le 2n$ at every *n* in the range. The lower line (Red) is $f_n = \lfloor n/2 \rfloor - 1$, while the top line (Red) is $f_n = n$. The L.H. figure is drawn in the range $4 \le n \le 113$. The R.H. figure is drawn in the range $4 \le n \le 5153$.

The number of elements of the set *E*, containing the collection of all distinct even numbers *e*, subject to $e, p_i, p_j \in \mathbb{N} \mid e$ is congruent to 0 (mod 2), $e = p_i + p_j$, $2 < p_i, p_j \le p_k$, strictly increases with increasing number of primes considered. These generated even numbers however, will not always be consecutive. If $e < p_k$ then obviously, we will obtain solutions with $p_i, p_j < p_k$. If however, $p_k < e \le 2pk$ in some pairs comprising the solution $e = p_i + p_j$, one of the primes say p_j, may and often will exceed p_k. The closer, some even number *e* is to $2p_k$, the greater the chance, that one of the summands will exceed p_k.

5. Deterministic Procedure Generating the Solution Set - The Methodology Of Approach

We may apply a simple procedure to generate the full solution set for a particular even number *e*:

Procedure 5.1 (Generating The Solution Set)

- Implement equations 4.4 through 4.6 above, to identify the particular variety, to which the even number *e* belongs.
- Compute *e* / 2. If *e* / 2 is prime, we may stop here, or proceed further to generate the entire solution set.
- Find the nearest to the *e* / 2, LPF or UPF integers (as is appropriate), one on each side of *e* / 2. In the case of *e* being of the Mixed Type, we have to run the procedure twice. We run the procedure once with the LPF integer being located on the low side and the second time with the LPF integer being on the high side. The UPF integer of course will always be located on the opposite side.
- If both of the found integers are prime numbers, they constitute a valid solution, possibly one of a number of valid solutions.
- Next, decrease the lower integer by 6 and increase the higher integer by 6. This constitutes another possible solution, providing that both are prime numbers.
- Continue to decrease/increase by 6 the respective integers, until the lower limit of 5 or 7 is reached. Every pair generated this way, constitutes a solution. However, to form a valid solution, both integers must simultaneously be prime.
- Verify if {3 , *e* - 3} constitutes a valid solution, if so, add it to the solution set. This solution is not categorized because, the prime number 3 is neither part of the **AP** LPF, nor UPF variety. The only **AP** LPF or UPF variety is *e* - 3 itself.

The procedure terminates upon reaching the predefined sentinel {5, 7, *e*} at the low or the high end, or both. In general, over the entire range of $2p_k$, there are two possible scenarios:

1) **Case A**: All distinct, consecutive even numbers up to and including $n = 2p_k$ were generated successfully. This implies that the function $N^{\circ} = p_k - 1$, (we obtain one even number less, because 2 does not equal $p_i + p_j$). Case A however, is an exception only.

2) **Case B**: In general, the smallest even number $e \in \mathbb{N}$ is found, which failed to be generated by the entire set $\{p_i, p_j \in \mathbb{N} \mid p_i , p_j \leq p_k\}$. The solution can be found by implementing the Procedure 5.1.

Case A. Due to the fact that the maximal gaps upper bound is given by Theorem 3.5:

$$(5.1) \quad \mathcal{UB} = \left\lfloor 5\left(\log_{10}\ p_k\right)^2 \right\rfloor \quad \forall p_k \ \in \ \mathbb{N} \mid p_k \geq 11$$

$$Equivalently, p_{(k+1)} - p_k \ \leq \ \mathcal{UB}$$

From inequality 5.1 we have that (upon relaxing the UB function by dropping the floor function):

$$p_{(k+1)} < \mathcal{UB} + p_k$$

and clearly,

$$\mathcal{UB} \ll p_k$$

Obviously therefore, we have that:

$$(5.2) \qquad p_{(k+1)} < 2p_k$$

Consequently, we see that we can generate all even numbers less than $p_{(k+1)}$ and more, up to and including $2p_k$. Five low p_k range cases are: {5, 7, 13, 19, 109}.

Case B. Suppose that some even number $e \in \mathbb{N}$ exists, such that $p_k < e < 2p_k$, which cannot be expressed as a sum $e = p_i + p_j \mid p_i, p_j \leq p_k$. We only need to be concerned with the smallest such number e. This is because, once the sequence of distinct and consecutive even integers is broken, it is impossible to generate this particular even number e, due to the fact that it does not have summands that both simultaneously $p_i, p_j \leq p_k$. Such summands constitute a legitimate solution, which cannot be reconciled with a failed generation of this number. At the very least it is a prime number $p = p_{(k+1)}$.

This even number e immediately terminates the generated sequence of distinct, consecutive even numbers. **All other even numbers, which possibly continue intermittently from this point on, up to the upper limit of $2p_k$, are not considered in this document**.

This number establishes a clear limit on the distinct, consecutive even numbers generated $e = p_i + p_j \mid p_i, p_j \leq p_k$. This will only change, after we reach the prime $p_{(k+c)}$ which actually is one of the summands of that number e. This implies, that such limit will possibly persist over a range of prime numbers p.

Clearly, the function N° remains constant in such a case until $p_{(k+c)}$. Consequently, the graph of the function N° exhibits a horizontal slope within such an interval. Therefore, by the definition N° is a weakly increasing function.

Now obviously, $e/2 \leq p_k < p_{(k+1)}$. By **Theorem 3.5**, we have that $p_{(k+1)} < 2p_k$. Since the set of primes $\{2, 3, 5, 7, \ldots, p_k\}$, does not form a solution $e = p_i + p_j$ we advance by implementing the Procedure 5.1. This implies that the next possible candidate for a valid summand (w.l.o.g. say p_j) in the range $p_k < p_j < e$ is $p_{(k+1)}$. This prime number must of course conform to the class variety code of e (equations 4.4 through 4.6).

This implies that the summand $p_j \geq p_{(k+1)}$. Since $p_j \geq p_{(k+1)}$, lets denote $p_j = p_{(k+c)}$ subject to $c \in \mathbb{N}$ in general. Then, the even number equals:

$$(5.3) \qquad e = p_{(k+c)} + p_i$$

This implies that,

$$(5.4) \qquad p_{(k+1)} < e < 2p_k$$

We can arrange the equations 4.4 through 4.6, to show their alignment:

$$(5.5) \qquad 6(m_1 + n_1 + 1) + 4 < 6(m_1 + n_1 + 1) + 5 < 6(m_2 + n_2 + 1) + 6 < 6(m_3 + n_3 + 1) + 7 < 6(m_3 + n_3 + 1) + 8$$

Frequently, some of the distinct variables m_i, or n_i will have the same numerical value. Referring however, to equations 4.4 through 4.6 above, we see that depending on e, they will originate from different variety of primes.

5.1 Examples

Example 5.2.

This example is analyzed carefully, providing all solutions. The number e = 224 cannot be formed by any combination of primes p_i , $p_j \leq p_k$ = 113 | $p_i + p_j$ = 224. This even number clearly is within the range of $2p_k$ = 226, however it requires a prime for one of the summands, which at that point is out of range. Consequently, the graph of the function N° exhibits a horizontal slope in the range 113 through 127 at the height of 108.

The complete solution for the even number e = 224 is presented in a tabular form. We implement the Procedure 5.1. This even number belongs to **Case 3**, which means that all UPF solution is required. Now, 224/2 = 112, hence we begin to search for possible UPF candidates at 112. The nearest lower candidate is 109, the nearest higher candidate is 115. Both are solutions to 224, however one of the summands is composite. Hence we proceed further decreasing/increasing the numbers. The results are listed in Table 3.

Table 2. Summands for e within the range 112 through 128:

Even number	Summands	e variety
112	5, 107	LPF
114	5, 109	Mixed
116	7, 109	UPF
118	59, 59	LPF
120	7, 113	Mixed
122	61, 61	UPF
124	11, 113	LPF
126	13, 113	Mixed
128	19, 109	UPF

Table 3. Summands comprising the solution of $p_i + p_j = 224$:

Summands	Solution type
109, 115	Invalid
103, 121	Invalid
97, 127	Valid
91, 133	Invalid
85, 139	Invalid
79, 145	Invalid
73, 151	Valid
67, 157	Valid
61, 163	Valid
55, 169	Invalid
49, 175	Invalid
43, 181	Valid
37, 187	Invalid
31, 193	Valid
25, 199	Invalid
19, 205	Invalid
13, 211	Valid
7, 217	Invalid

Example 5.3.

The even number *e* = 4952, cannot be formed from the set of primes 3, ... , 2539. The upper limit for this prime is 2 ∗ 2539 = 5078, hence, 4952 is well within this range. This particular even number *e*, constitutes **Case 3**, all UPF primes solution. Searching systematically, we find that *e* = 4952 = 2293 + 2659. The prime 2659, is the 6-th UPF prime after the prime 2539. The full set is:

{2539, 2543, 2549, 2551, 2557, 2579, 2591, 2593, 2609, 2617, 2621, 2633, 2647, 2657, 2659}

The LPF primes are in Black, while the UPF primes are in Red.

6. Resolution of the Goldbach Conjecture

We begin with some definitions and preliminary derivations.

Definition 6.1 (Interval length)**.**

$$c = \lfloor 5\sqrt{p_k} \rfloor \quad \forall p_k \in \mathbb{N} \mid p_k \geq 5$$

Definition 6.2 (Interval endpoint)**.**

$$t = (p_k + c)$$

Definition 6.3 (Product of primes within an interval)**.**

$$p_L\sharp = \prod_{p_k < p_i \leq t} (p_i) \quad s.t.\ p_i = 6s + 5 \qquad\qquad p_U\sharp = \prod_{p_k < p_i \leq t} (p_i) \quad s.t.\ p_i = 6s + 7$$

The products of primes $p_L\#$ and $p_U\#$ in the Definition 6.3, are defined to equal 1 in the case that there are no pertinent primes within the interval. From the Definitions 2.4, 2.5 and inequality 3.1 we derive:

$$(6.1)\quad \mathcal{LB}_t \leq \log \mathcal{M}_t = \log\left(\frac{(p_k + c)!}{(p_k! \times c!)}\right) = \sum_{p \leq t} \mathcal{K}_p \log p \leq \mathcal{UB}_t \quad \forall p_k \in \mathbb{N} \mid p_k \geq 5$$

Where p is as usual a prime number. We derive the bounds on the logarithm of M_L from Lemmas 2.7 and 2.8 and from the Definition 6.3 (**case : LPF primes**):

$$(6.2) \quad \mathcal{LB}_L = \left(\log \left(\frac{(t+1)^{(t+1)}}{(p_k+1)^{(p_k+1)} \ (c+1)^{(c+1)} \, (p_U\sharp)} \right) \right)$$

$$\leq \log \mathcal{M}_L = \sum_{k=1}^{c} \log (p_k + k) - \sum_{k=1}^{c} \log k - \sum_{p_k < p_U}^{t} \log p_U$$

$$\leq \log \left(\frac{t^t}{p_k^{p_k} \ c^c \, (p_U\sharp)} \right) - 1 = \mathcal{UB}_L \quad \forall p_k \in \mathbb{N} \mid p_k \geq 5$$

The proof of the Solution Bound Theorem implements the Supremum bound function UB_{Ls}. This means that all **AP** UPF primes within the interval $p_k < p_U \leq t$ are factored out from log M_t function and its bounds. Due to the fact that the Supremum function UB_L applies values of p_k, c and t directly, it imposes a technical requirement to generate a set of pertinent values, to correctly approximate the interval s. This is to ascertain that the generated interval is at least equal to s, and the corresponding value of c is correct. Respective definitions follow:

Definition 6.4

$$p_{k_s} = \frac{p_k}{2}$$

Definition 6.5

$$c_s = \left\lfloor 5\sqrt{(p_{k_s})} \right\rfloor$$

Definition 6.6

$$t_s = p_{k_s} + c_s$$

Theorem 6.7 (Bound on **AP** primes within an interval).

*For any $p_k \in \mathbb{N} \mid p_k \geq 5$, there exists at least one **AP** LPF/UPF prime number $p_j \in \mathbb{N} \mid p_k < p_j \leq t$ The upper bound $UB_{(pk)}$ is given by:*

$$(6.3) \quad c = \mathcal{UB}_{p_k} = \lfloor 5\sqrt{p_k} \rfloor \qquad \forall p_k \in \mathbb{N} \mid p_k \geq 5$$

$$Equivalently, p_j - p_k \leq \mathcal{UB}_{(p_k)}$$

Remark 6.1. The proof considers the AP LPF primes only. The case of AP UPF primes is analogous, thus it is left as an exercise for the reader (need to replace the $p_U\#$ with $p_L\#$).

Proof.

Suppose that there is no **AP** LPF prime within the interval from p_k to t for $p_k \in \mathbb{N} \mid p_k \geq 199$. Then in accordance with the hypothesis, by Lemma 3.1 we have that, every prime factor p of $M_{(t)}$ (the UPF primes were a priori factored out: the term $p_U\#$) must be less than or equal to $s = \lfloor t/2 \rfloor$. Invoking Definitions 6.4, 6.5, 6.6, and the inequality 6.1, we derive for all $p_k \in \mathbb{N} \mid p_k \geq 199$:

$$(6.4) \quad \mathcal{LB}_L = \log\left(\frac{(t+1)^{(t+1)}}{(p_k+1)^{(p_k+1)}\,(c+1)^{(c+1)}\,(p_U\sharp)}\right)$$

$$\leq \log \mathcal{M}_L = \sum_{p \leq t_{(s)}} \mathcal{K}_p \log p - \sum_{p_k < p_U}^{t} \log p_U \leq \log\left(\frac{(t_s)^{t_s}}{(p_{k_s})^{p_{k_s}}\,(c_s)^{c_s}\,(p_U\sharp)}\right) - 1 = \mathcal{UB}_{L_s}$$

In accordance with the hypothesis therefore, it must be true that:

$$(6.5) \qquad a_c = \log\left(\frac{(t+1)^{(t+1)}}{(p_k+1)^{(p_k+1)}\,\,(c+1)^{(c+1)}}\right) - \log\left(\frac{(t_s)^{t_s}}{(p_{k_s})^{p_{k_s}}\,\,(c_s)^{c_s}}\right) + 1 < 0$$

Now, we apply the Cauchy's Root Test for $p_k \geq 199$:

$$(6.6) \qquad \lim_{c\to\infty} \sqrt[c]{|a_c|} = \lim_{c\to\infty} \sqrt[c]{|\mathcal{LB}_L - \mathcal{UB}_{L_s}|}$$

At $p_k = 199$ the Cauchy's Root Test attains approx. 1.05921 and tends asymptotically to 1, decreasing strictly from above. Thus, by the definition of the Cauchy's Root Test, the series formed from the terms of the difference $LB_{(t)}$ - $UB_{(ts)}$, diverges as p_k increases unboundedly. Hence in accordance with the hypothesis, inequality 6.5 diverges to negative infinity as p_k increases unboundedly. However, at $p_k = 199$ the difference 6.5 attains approx. 56.0882 and diverges as p_k increases unboundedly. Hence, we have a contradiction to the initial hypothesis. This implies that for all $p_k \in \mathbb{N} \mid p_k \geq 199$:

$$(6.7) \qquad \mathcal{LB}_t - \mathcal{UB}_{t_s} \;>\; 0$$

Necessarily therefore, there must be at least one **AP** LPF prime within the interval c for any $p_k \in \mathbb{N} \mid p_k \geq 199$. Table 4 (in the Appendix) lists all pertinent primes within the interval $p_k \in \mathbb{N} \mid 5 \leq p_k \leq 199$, n $\leq$ t. Thus we deduce that, Theorem 6.7 holds in this range as well. Consequently, Theorem 6.7 holds as stated for all $p_k \in \mathbb{N} \mid p_k \geq 5$, hence completing the proof.

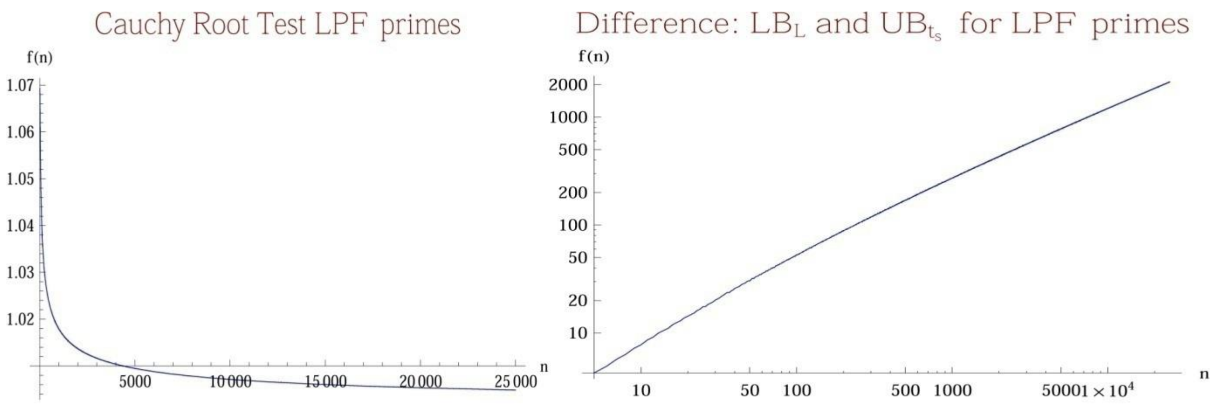


Figure 3. The L.H. drawing shows the Cauchy's Root Test. The R.H. log-log drawing shows the difference of the LB_L - $UB_{(Ls)}$. The figures are drawn at every $n \in \mathbb{N} \mid 5 \leq n \leq$ 25000.

Definition 6.8. $c = n = e/2$

Definition 6.9. $t = n + c = 2n$

Definition 6.10 (Product of primes in the interval)

$$p_S\sharp = \prod_{p_i \le n} (p_i) \qquad \text{for } p_i = (e - p_j)\,,\ \text{and } p_i \le p_j$$

In the case that there are no prime summands in the interval, $p_S\#$:= 1. It is necessary and sufficient for the product of primes $p_S\#$, to take into account the smaller prime of the solution pair only.

Theorem 6.11 (The Binary/Strong Goldbach Conjecture)**.**
The exist at least one solution to $e = p_i + p_j \mid e, p_i, p_j \le 2n$, for $n \in \mathbb{N} \mid n = e/2$ *and* $p_i \le p_j$. *This implies that any number e* congruent to 0(mod 2) | $e \ge 4$ *can be formed as a sum of two prime numbers. The combination of primes that comprise the solution is not unique.*

Proof.
Suppose that for some $n \in \mathbb{N} \mid n \ge 227$, $n = e/2$, there is no solution to

$$e = p_i + p_j \mid pi \le e/2 \le p_j$$

From the Definitions 2.4, 2.5 and inequality 3.1, as well as Definitions 6.8 and 6.9 we derive for all $n \in \mathbb{N} \mid n \ge 227$:

$$(6.8) \quad \mathcal{LB}_t = \log\left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)}\,(c+1)^{(c+1)}}\right) \le \log \mathcal{M}_t = \sum_{p\le t} \mathcal{K}_p \log p = \log\left(\frac{t!}{(n!)\ (c!)}\right)$$

Since for all $n \in \mathbb{N} \mid n \ge 227$ we have $n > 2\,(5\sqrt{n})$ therefore, by Theorem 6.7 multiple p_j exist within the interval of length $c = n \mid n \le p_j < t$. Thus necessarily, in accordance with the hypothesis, this implies that not a single $p_i = e - p_j$ exists. Consequently, in accordance with the Definition 6.10:

$$(6.9) \qquad p_S\sharp = \prod_{p_j \le t} (e - p_j) = 1$$

From inequality 6.8 and equation 6.9 we have:

$$(6.10) \qquad a_c = \log\left(\frac{(t+1)^{(t+1)}}{(n+1)^{(n+1)}\ \ (c+1)^{(c+1)}}\right) - \log\left(\frac{t!}{(n!)\ (c!)\,(p_S\sharp)}\right) < 0$$

Now, we apply the Cauchy's Root Test:

$$(6.11) \qquad \lim_{c\to\infty} \sqrt[c]{|a_c|} = \lim_{c\to\infty} \sqrt[c]{\left|\mathcal{LB}_t - \log\left(\frac{t!}{(n!)\ (c!)\,(p_S\sharp)}\right)\right|}$$

At n = 227 the Cauchy's Root Test attains approx. 1.01722 and tends asymptotically to 1, decreasing strictly from above. Please also refer to Figure 5. Thus, by the definition of the Cauchy's Root Test, the sequence formed from the terms of the difference $LB_{(t)}$ - log M_t, diverges as n increases unboundedly. Hence in accordance with the hypothesis, inequality 6.10 diverges to negative infinity as n increases unboundedly. However, at n = 227 the difference 6.10 attains approx. 42.8228 and diverges as n increases unboundedly. Please also refer to Figure 5. Hence, we have a contradiction to the initial hypothesis. This implies that for all $n \in \mathbb{N} \mid n \ge 227$:

$$(6.12) \qquad \mathcal{LB}_t - (\log \mathcal{M}_t - \log(p_S\sharp)) \; > \; 0$$

Necessarily therefore, there must be at least one prime solution within the interval *c* for all $n \in \mathbb{N} \mid n \geq 227$. Table 5 in the Appendix lists the number of solutions at any $e \in \mathbb{N} \mid 2 \leq e \leq 468$. Figure 4 shows the graph of the number of existing solutions at every even number within the range. Evidently, every possible even number $e \in \mathbb{N} \mid 2 \leq e \leq 468$ is satisfied. Thus we deduce that Theorem 6.11 holds in this range as well. Consequently Theorem 6.11 holds as stated for all $e \in \mathbb{N} \mid e \geq 4$, thus completing the proof.

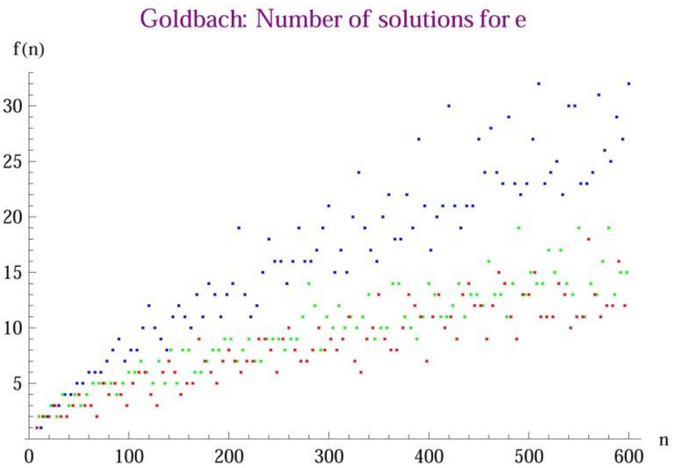


Figure 4. The drawing shows the number of existing solutions of $e = p_i + p_j$ at every even number $e \in \mathbb{N} \mid 8 \leq e \leq 600$.

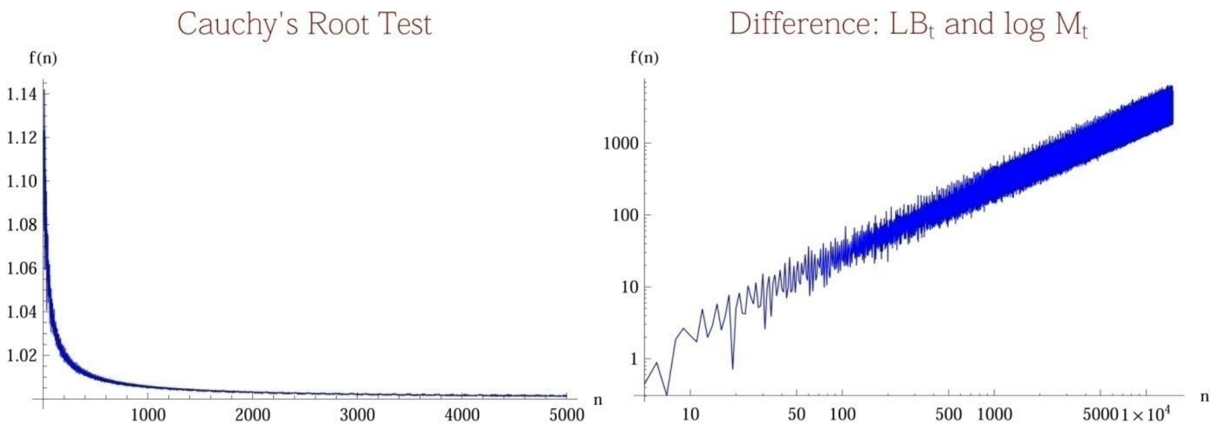


Figure 5. The L.H. drawing shows the Root Test. The R.H. log-log drawing shows the difference 6.10. It is drawn at every $n \in \mathbb{N} \mid 5 \leq n \leq 15000$.

Figure 4 displays the so called Goldbach's comet. The Figure 6 in the Appendix shows an enhanced version of that graph. The color coding implemented in both Figures denotes:

- Even numbers $e \in \mathbb{N} \mid e = 6m + 4$ comprised of All LPF solutions in Green.
- Even numbers $e \in \mathbb{N} \mid e = 6m + 6$ comprised of Mixed solutions in Blue.
- Even numbers $e \in \mathbb{N} \mid e = 6m + 8$ comprised of All UPF solutions in Red.

The number of all LPF or all UPF solutions to $e = p_i + p_j$ varies, it does so however within common for LPF/UPF boundaries. Clearly, the number of the Mixed solutions to

$e = p_i + p_j$ exceeds that of the LPF or UPF solutions. The reason of that is that there are essentially two solution sets in the Mixed case. One set is comprised of solutions to
$e = p_i + p_j \mid p_i \in LPF$; while $p_j \in UPF$ and the second set of solutions is $e = p_i + p_j \mid p_i \in UPF$, while $p_j \in LPF$. This at times results in almost doubling the solution set, while in the remaining instances it maintains the solution set significantly larger than either LPF/UPF. Hence creating the bands of solutions on the graph.

Corollary 6.12 (The Ternary Goldbach Conjecture)**.**
The ternary Goldbach conjecture asserts that *for all* $s \in \mathbb{N} | s$ being congruent to 1(mod 2) and $s \geq 7$, s can be written as the sum of three prime numbers.

Proof.

By Theorem 6.11 the Binary Goldbach Conjecture holds, for all $e \in \mathbb{N} | \ e \geq 4$. Therefore, any arbitrary integer $e \in \mathbb{N} | \ e$ congruent to 0(mod 2) and $e \geq 4$ can be written as a sum of two prime numbers. Corollary 6.12 clearly holds in the case of $p = 7$ as $7 = 2 + 2 + 3$. Now, an arbitrary $s \in \mathbb{N} | \ s$ congruent to 1(mod 2), $s > 7$, may also be written as $s = e + 3$. Because e is an arbitrary even integer, which may be written as the sum of exactly two primes, necessarily therefore s, which is an arbitrary odd integer, may be written as the sum of three primes. Hence Corollary 6.12 holds as stated for all $s \in \mathbb{N} | \ s$ congruent to 1(mod 2) and $s \geq 7$, concluding the proof.

The following Corollary 6.13 resolves a question posed by Shi et al (2019).

Corollary 6.13(Representation of Even Numbers by the difference of two primes)
Any even number $e \in \mathbb{N} \cup \{0\}$ represents the difference between two prime numbers.

Proof.

By Theorem 6.11 the Binary Goldbach Conjecture holds, for all $e \in \mathbb{N} | \ e \geq 4$. Therefore, any arbitrary integer $e \in \mathbb{N} | \ e$ congruent to 0(mod 2) can be written as:

$$(6.13) \qquad p_i + p_j = e = 2s, \ \forall s \in \mathbb{N} \mid s \geq 2 \ \text{ and } p_i < p_j$$

Obviously,

$$(6.14) \qquad (p_i + p_j) - p_i = p_j$$

Consequently,

$$(6.15) \qquad (p_i + p_j) - 2p_i = p_j - p_i$$

This implies that

$$(6.16) \qquad e - 2p_i = 2s - 2p_i = 2\,(s - p_i) = p_j - p_i$$

Which by Theorem 6.11 holds for any two prime numbers $p_i, p_j \in \mathbb{N} | p_i + p_j = e$, s.t. $e \in \mathbb{N} | e$ congruent to 0(mod 2); $e \geq 4$. In case of Twin Primes we have $(p_j + p_i) - 2\,(p_i)$ which equals 2 in this case; with $p_j = p_i + 2$. In the case of a double of any prime we have : $(p_j + p_i) - 2\,(p_i) = 0$; with $p_j = p_i$. Thus Corollary 6.13 holds in these cases as well. Necessarily, we deduce that Corollary 6.13 holds as stated, concluding the proof.

From equations 4.4 through 4.6, there are two cases to consider when contemplating the difference of two prime numbers. These cases generate two disjoint sets of even numbers. Firstly, all LPF or all UPF difference collapses into one case, while the Mixed case produces 2 instances;

- **Case 1**: The even number *e* is all LPF or all UPF variety. The difference of two prime numbers p_i, $p_j \in \mathbb{N} | p_i < p_j$ produces an even integer of the form:

$$e = (6m + C) - (6n + C) = 6\,(m - n)\,, \text{ generating the set } \{6, 12, 18, 24, \ldots\} \tag{6.17}$$

- **Case 2**: The even number *e* is of the Mixed variety. The difference of an arbitrary LPF integer and an arbitrary UPF integer (or vice versa) produces an even integer of the form:

$$e = (6m + C_2) - (6n + C_1) = 6\,(m - n) - (C_2 - C_1) \tag{6.18}$$
$$\text{with } (C_2 - C_1) = \pm 2, \text{ generating the set } \{4, 8, 10, 14, \ldots\}$$

It is obvious that the union of the two solution sets given above by Equations 6.17 and 6.18, produces the whole set of the even integers $e \in \mathbb{N} | e \geq 4$.

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Appendix

Table 4. Bound on AP Primes: Theorem 6.7

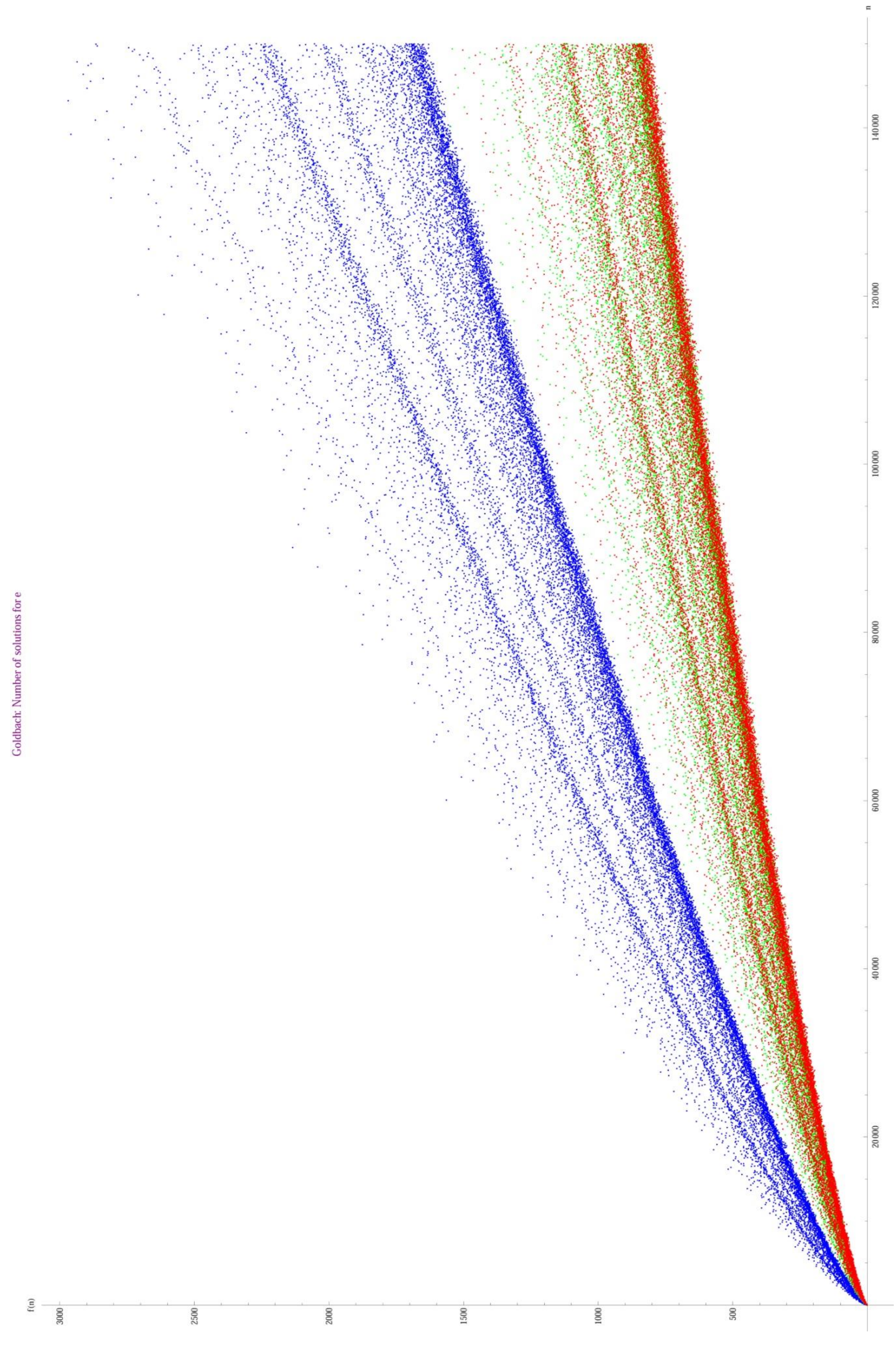
The Set of Primes within interval $\mathcal{UB}_{p_k}$		
p_k	$\mathcal{UB}_{p_k}$	Set of LPF Primes
5	11	$\{11\}$
7	13	$\{11, 17\}$
11	16	$\{17, 23\}$
13	18	$\{17, 23, 29\}$
17	20	$\{23, 29\}$
19	21	$\{23, 29\}$
23	23	$\{29, 41\}$
29	26	$\{41, 47, 53\}$
31	27	$\{41, 47, 53\}$
37	30	$\{41, 47, 53, 59\}$
41	32	$\{47, 53, 59, 71\}$
43	32	$\{47, 53, 59, 71\}$
47	34	$\{53, 59, 71\}$
53	36	$\{59, 71, 83, 89\}$
59	38	$\{71, 83, 89\}$
61	39	$\{71, 83, 89\}$
67	40	$\{71, 83, 89, 101, 107\}$
71	42	$\{83, 89, 101, 107, 113\}$
73	42	$\{83, 89, 101, 107, 113\}$
79	44	$\{83, 89, 101, 107, 113\}$
83	45	$\{89, 101, 107, 113\}$
89	47	$\{101, 107, 113, 131\}$
97	49	$\{101, 107, 113, 131, 137\}$
101	50	$\{107, 113, 131, 137, 149\}$
103	50	$\{107, 113, 131, 137, 149\}$
107	51	$\{113, 131, 137, 149\}$
109	52	$\{113, 131, 137, 149\}$
113	53	$\{131, 137, 149\}$
127	56	$\{131, 137, 149, 167, 173, 179\}$
131	57	$\{137, 149, 167, 173, 179\}$
137	58	$\{149, 167, 173, 179, 191\}$
139	58	$\{149, 167, 173, 179, 191, 197\}$
149	61	$\{167, 173, 179, 191, 197\}$
151	61	$\{167, 173, 179, 191, 197\}$
157	62	$\{167, 173, 179, 191, 197\}$
163	63	$\{167, 173, 179, 191, 197\}$
167	64	$\{173, 179, 191, 197, 227\}$
173	65	$\{179, 191, 197, 227, 233\}$
179	66	$\{191, 197, 227, 233, 239\}$
181	67	$\{191, 197, 227, 233, 239\}$
191	69	$\{197, 227, 233, 239, 251, 257\}$
193	69	$\{197, 227, 233, 239, 251, 257\}$
197	70	$\{227, 233, 239, 251, 257, 263\}$
199	70	$\{227, 233, 239, 251, 257, 263, 269\}$
The End		

Table 5. The Binary Goldbach Conjecture: Theorem 6.11

Number of solutions of $e = p_i + p_j \mid e \equiv 0 (\mathrm{mod}\ 2)$					
e	Solutions	e	Solutions	e	Solutions
4	1	158	5	314	9
6	1	160	8	316	10
8	1	162	10	318	15
10	2	164	5	320	11
12	1	166	6	322	11
14	2	168	13	324	20
16	2	170	9	326	7
18	2	172	6	328	10
20	2	174	11	330	24
22	3	176	7	332	6
24	3	178	7	334	11
26	3	180	14	336	19
28	2	182	6	338	9
30	3	184	8	340	13
32	2	186	13	342	17
34	4	188	5	344	10
36	4	190	8	346	9
38	2	192	11	348	16
40	3	194	7	350	13
42	4	196	9	352	10
44	3	198	13	354	20
46	4	200	8	356	9
48	5	202	9	358	10
50	4	204	14	360	22
52	3	206	7	362	8
54	5	208	7	364	14
56	3	210	19	366	18
58	4	212	6	368	8
Continued ...					

Solutions (Continued)					
60	6	214	8	370	14
62	3	216	13	372	18
64	5	218	7	374	10
66	6	220	9	376	11
68	2	222	11	378	22
70	5	224	7	380	13
72	6	226	7	382	10
74	5	228	12	384	19
76	5	230	9	386	12
78	7	232	7	388	9
80	4	234	15	390	27
82	5	236	9	392	11
84	8	238	9	394	11
86	5	240	18	396	21
88	4	242	8	398	7
90	9	244	9	400	14
92	4	246	16	402	17
94	5	248	6	404	11
96	7	250	9	406	13
98	3	252	16	408	20
100	6	254	9	410	13
102	8	256	8	412	11
104	5	258	14	414	21
106	6	260	10	416	10
108	8	262	9	418	11
110	6	264	16	420	30
112	7	266	8	422	11
114	10	268	9	424	12
116	6	270	19	426	21
118	6	272	7	428	9
120	12	274	11	430	14
122	4	276	16	432	19
124	5	278	7	434	13
126	10	280	14	436	11
128	3	282	16	438	21
130	7	284	8	440	14
132	9	286	12	442	13
134	6	288	17	444	21
136	5	290	10	446	12
138	8	292	8	448	13
140	7	294	19	450	27
142	8	296	8	452	12
144	11	298	11	454	12
Continued . . .					

Solutions (Continued)					
146	6	300	21	456	24
148	5	302	9	458	9
150	12	304	10	460	16
152	4	306	15	462	28
154	8	308	8	464	12
156	11	310	12	466	13
—	—	312	17	468	24
The End					


Goldbach: Number of solutions for e
f(n)
n
3000
2500
2000
1500
1000
500
20 000
40 000
60 000
80 000
100 000
120 000
140 000